A HIGHER DIMENSIONAL GENERALIZATION OF TAUT FOLIATIONS

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Abstract. A higher dimensional generalization of taut foliations is introduced. Tools from symplectic geometry are used to describe surgery constructions, and to study the space of leaves of this class of foliations.

Codimension one foliations are too large a class of structures to obtain strong structure theorems for them. According to a theorem of Thurston [39] a closed manifold admits a codimension one foliation if and only if its Euler characteristic is vanishing. In order to draw significant results it is necessary to assume the existence of new structures compatible with the foliation. We mention two possible approaches (in what follows the manifolds will always be closed and oriented, the codimension one foliations co-oriented, and all the structures and maps smooth unless otherwise stated):

(1) *Impose the existence of a richer transversal structure.* The foliation on M is defined by charts $\varphi_i\colon \mathbb{R}^p \times \mathbb{R} \to M$ sending the leaves $\mathbb{R}^p \times \{\cdot\}$ to the leaves on $\mathcal{F}$ (*charts adapted* to $\mathcal{F}$). Let $\mathcal{T}$ be the disjoint union of the transversals $\{0\} \times \mathbb{R}$; it carries a smooth structure. The change of coordinates generate a pseudogroup Γ of transformations of $\mathrm{Diff}(\mathcal{T})$, the *holonomy pseudogroup*. A transversal structure on $\mathcal{F}$ is a structure on $\mathcal{T}$ invariant by the action of Γ. For example we have transversely analytic foliations, whose existence prevents $\pi_1(M)$ from being finite [16]; riemannian foliations (when Γ are isometries of some riemannian metric on $\mathcal{T}$), for which there is a structure theorem [28] implying that $\mathcal{F} = \ker\alpha$, with $d\alpha = 0$, and therefore M is a fiber bundle over S^1; foliations with a transversal invariant measure (when Γ are ergodic w.r.t some measure on $\mathcal{T}$), whose existence -related to the growth of the leaves- has strong consequences on the topology of M [33].

(2) *Introduce metrics (resp. closed forms) adapted to the foliation in some sense.* Once we fix a metric in $(M^{p+1}, \mathcal{F})$, there is an induced *comass norm* $||\cdot||^*$ on p-forms [17]. A p-current T is a current of integration if its mass $M(T) := \sup\{T(\alpha), ||\alpha||^* \leq 1\}$ is finite. Any integration current has an associated local Radon measure $||T||$, and for $||T||$-a.e. point x in M there is an associated measurable field of p-vectors T_x such that

$$T(\alpha) = \int \alpha(T_x) d||T||(x),\ \forall \alpha \in \Omega^p(M)$$

T is said to be a foliation current if $T_x = F_x$ for $||T||$-a.e. x in M, where F_x is the oriented unit p-vector spanning the tangent space to $\mathcal{F}$ at x. Closed foliation currents are in one to one correspondence with transversal invariant measures.

$\mathcal{F}$ is said to be geometrically tight if M carries a metric such that every foliation current is mass minimizing among its cohomologous integrable currents. Geometric tightness has interesting consequences regarding the

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growth of the leaves of the foliation and the existence of compact leaves [17].

We would like to take a closer look at the second approach. We will use from now on the subscript $\mathcal{F}$ (resp. W if W is a submanifold of M) to denote the restriction of a form, connection,... to the leaves of $\mathcal{F}$ (resp. to W).

Recall that geometric tightness is characterized by the existence a closed p-form ξ which is positive on $\mathcal{F}$ [17] (and which is a calibration for $\mathcal{F}$ w.r.t some metric g [18]). The kernel of ξ is a line field transversal to $\mathcal{F}$. Let X be any no-where vanishing vector field in kerξ. Then $\mathcal{L}_X\xi = 0$ and if X (possibly locally defined) preserves $\mathcal{F}$ we also have $\mathcal{L}_X\xi_{\mathcal{F}} = 0$. Hence, geometric tightness can be understood as the existence of a leafwise volume form which for an appropriate transversal direction remains unchanged.

Locally we can take charts adapted to $\mathcal{F}$ with coordinates $x_1, \dots, x_p, x_{p+1}$, and make sure that the line field kerξ goes to "vertical" line field spanned by $\partial/\partial x_{p+1}$. We can always find a self-diffeomorphism of the leaf through the origin so that $\xi_{\mathcal{F}}$ restricted to that leaf is pulled back to

$$\Xi_{\mathbb{R}^p} := dx_1 \wedge \cdots \wedge dx_p$$

If we extend it independently of the vertical coordinate x_{p+1} then ξ is pulled back to

$$\xi_{\mathbb{R}^{p+1}} := dx_1 \wedge \cdots \wedge dx_p \in \Omega^p(\mathbb{R}^{p+1})$$

Therefore, geometric tightness is equivalent to the existence of a reduction of the structural pseudogroup of $(M, \mathcal{F})$ to $\mathrm{Vol}(\mathbb{R}^p, \Xi_{\mathbb{R}^p}) \times \mathrm{Diff}(\mathbb{R})$, where $\mathrm{Vol}(\mathbb{R}^p, \Xi_{\mathbb{R}^p})$ denotes the pseudogroup of diffeomorphisms (defined on open sets) of $\mathbb{R}^p$ preserving the volume form $\Xi_{\mathbb{R}^p}$.

From this point of view it is easy to see how to construct closed transversal cycles through any point: for each $x \in M$ parametrize the orbit of kerξ through x with g-speed 1 and in the positive direction (we have a fixed riemannian metric g). For each $\epsilon > 0$ small enough, let $B_{\mathcal{F}}(x, \epsilon)$ be the ball of radius ϵ in the leaf through x and centered at x. Let $\varphi_x \colon B_{\mathcal{F}}(x, \epsilon) \times [0, \infty) \to M$ be the one parameter family of diffeomorphisms defined by integrating a (local) vector field in kerξ such that the diffeomorphisms preserve the leaves, and normalized by declaring $\varphi_x(x, t)$ to be the aforementioned parametrization of the orbit through x. By compactness, there exists a constant $C > 0$ such that the volume swept out in time t is greater than $tC\mathrm{Vol}(B_{\mathcal{F}}(x, \epsilon))$. Again by compactness there exists a $t_\epsilon > 0$ such that $\varphi_{x,t_\epsilon}(B_{\mathcal{F}}(x, \epsilon)) \cap B_{\mathcal{F}}(x, \epsilon) \neq \emptyset$ (this is essentially a classical argument that goes back to Poincarè). By deflecting a bit the orbit for values of t in $[t_\epsilon - \delta, t_\epsilon]$, for δ arbitrarily small one constructs closed cycles (see [38], where this elementary construction appears).

Each time that the orbit enters $B_{\mathcal{F}}(x, \epsilon)$ we have a return map which belongs to the pseudogroup $\mathrm{Vol}(\mathbb{R}^p, \Xi_{\mathbb{R}^p})$. If $p = 2$ -i.e. if we have a taut foliation in a 3-manifold- then under certain circumstances we can deduce interesting geometric information about the existence of more closed orbits (Poincaré-Birkhoff theorem). If $p > 2$ we have little "geometric" control on the return map, for the only invariant -assuming its domain to be diffeomorphic to a ball- is the total volume [15], and problems as the existence of higher dimensional transversal submanifolds seem difficult to attack.

It has been known for some time that if one wants higher dimensional generalizations of the Poincaré-Birkhoff theorem, then the right setting is not volume geometry but symplectic geometry [19]. Recall that a symplectic structure on a

manifold X^{2n} (always of even dimension) is a closed 2-form Ω which is maximally non-degenerate, i.e. Ω^n is a volume form.

We want to propose the following higher dimensional generalization of taut foliations (see also [22]):

Definition 1. *A (co-oriented) codimension one foliation $\mathcal{F}$ of M^{2n+1} is said to be 2-calibrated if there exists a closed 2-form ω such that ${\omega_{\mathcal{F}}}^n$ is no-where vanishing. The 2-calibrated foliation is integral if $[\omega/2\pi] \in H^2(M;\mathbb{Z})$.*

Our main result is:

Theorem 1. *Let $(M, \mathcal{F}, \omega)$ be a 2-calibrated foliation. Then there exists an embedding $W^3 \hookrightarrow M$ such that*

1. *W^3 is transversal to $\mathcal{F}$ and ω_W calibrates $\mathcal{F}_W$. Hence, $(W^3, \mathcal{F}_W)$ is a taut foliation.*
2. *The inclusion $(W^3, \mathcal{F}_W) \hookrightarrow (M, \mathcal{F})$ descends to a homeomorphism of leaf spaces $W/\mathcal{F}_W \to M/\mathcal{F}$.*

Point 1 in the above result can be thought as a manifestation of the fact that the return maps associated to the flow of kerω belong to $\mathrm{Symp}(\mathbb{R}^{2n}, \Omega_{\mathbb{R}^{2n}})$, the pseudogroup of diffeomorphism (with domain an open set) of $\mathbb{R}^{2n}$ that preserve the canonical symplectic structure

$$\Omega_{\mathbb{R}^{2n}} := \sum_{i=1}^{n} dx_i \wedge dy_i \tag{1}$$

These *symplectomorphisms* are much more rigid than the transformations preserving the volume form $\Omega^n_{\mathbb{R}^{2n}} = \Xi_{\mathbb{R}^{2n}}$ (see for example [19]). They preserve the symplectic invariants of the corresponding subsets of $\mathbb{R}^{2n}$, so for example these cannot be squeezed along symplectic 2-planes (the symplectic capacities have to be preserved); naively, one might try to construct the transversal 3-manifolds of theorem 1 by choosing tiny 2-dimensional symplectic pieces Σ_x inside a leaf, whose image by the return map (for very large time, because the surfaces are very tiny) is a small 2-dimensional symplectic manifold $\varphi_{x,t_\epsilon}(\Sigma_x)$ that can be isotoped to Σ_x through symplectic surfaces. Then the isotopy could be used to connect $\varphi_{x,t_\epsilon-\delta}(\Sigma_x)$ with Σ_x and thus get a piece of 3-dimensional taut foliation. Of course, this idea seems difficult to be carried out, for the different pieces should be combined to construct a closed manifold, but it gives some insight of why point 1 in theorem 1 holds true. Actually this result is a rather elementary consequence of the approximately holomorphic techniques for symplectic manifolds introduced by Donaldson [7, 9]. The proof of point 2 in theorem 1, however, is more elaborate and uses the Lefschetz pencil structures for 2-calibrated foliations introduced in [22].

The paper is organized as follows. In section 1 we introduce basic facts and definitions about 2-calibrated foliations and give examples. We also study their relation with regular Poisson structures.

In order to show that 2-calibrated foliations are a broad enough class of foliations, section 2 presents a surgery construction modelled on the normal connected sum for symplectic manifolds. A surgery construction based on generalized Dehn twists is presented in section 3. In subsection 3.4 this *generalized Dehn surgery* is shown to admit an equivalent description as the new end of a cobordism which amounts to attaching to the trivial cobordism a symplectic handle along a parametrized lagrangian sphere; to prove the equivalence we need some results about (i) cosymplectic structures and about (ii) the symplectic monodromy around a critical value of a symplectic fibration. As a byproduct, we get a proof of a result announced

by Giroux and Mohsen [13] relating generalized Dehn surgery along a parametrized lagrangian sphere in an open book compatible with a contact structure, and legendrian surgery along the aforementioned sphere.

In section 4 we recall the notion of a Lefschetz pencil structure for $(M, \mathcal{F}, \omega)$ and the main existence theorem. A Lefschetz pencil structure admits a leafwise symplectic connection. The basic facts about the parallel transport are collected in subsection 4.1. They are a basic tool to relate (i) the leaf space of a regular fiber of the pencil with the leaf space of $(M, \mathcal{F}, \omega)$ and (ii) the 2-calibrated foliations induced on different regular fibers. In subsection 4.2 it is proven that the 2-calibrated foliations that inherit any two regular fibers of the pencil are related by a sequence of generalized Dehn surgeries. In subsection 4.3 we point out the relation of the growth types of the leaves of M and of the fibers.

In section 5 a Lefschetz pencil structure for a 3-dimensional taut foliation is seen to decompose M^3 –away from a well behaved measure zero set– as the disjoint union of solid tori $T^2 \times S^1$ whose S^1-fibers are transversal cycles. This decomposition is used to define in a geometric way families of harmonic measures with full support.

1. Definitions and examples

Definition 2. *Let $(M, \mathcal{F}, \omega)$ be a 2-calibrated foliation and $l \colon N \hookrightarrow M$ a submanifold. N is a 2-calibrated submanifold if $(N, l^*\mathcal{F}, l^*\omega)$ is a 2-calibrated foliation.*

The definition of a 2-calibrated foliation can be given locally.

Definition 3. *A 2-calibration for $(M, \mathcal{F})$ is a reduction of its structural pseudogroup to Sympl$(\mathbb{R}^{2n}, \Omega_{\mathbb{R}^{2n}}) \times$ Diff$(\mathbb{R})$.*

To prove that definitions 1 and 3 are equivalent we need the following Darboux type result:

Lemma 1. *Let $x_1, y_1, \ldots, x_n, y_n, t$ be coordinates on $\mathbb{R}^{2n+1}$ and let*

$$\omega_{\mathbb{R}^{2n+1}} := \sum_{i=1}^{n} dx_i \wedge dy_i$$

Around any $x \in M$ we can find a chart $\varphi \colon (\mathbb{R}^{2n} \times \mathbb{R}, 0) \to (V, x)$ adapted to $\mathcal{F}$ and such that $\varphi^*\omega = \omega_{\mathbb{R}^{2n+1}}$.

Proof. We start with a chart centered at x and adapted to $\mathcal{F}$. Next we modify it (preserving all leaves setwise) so that the kernel of ω is sent to the "vertical" lines $x_1 = c_1, y_1 = c_2, \ldots, x_n = c_{2n-1}, y_n = c_{2n}$. Finally we apply Darboux' lemma to the leaf through the origin. The relevant observation is that the kernel of ω matching the vertical lines together with ω being closed implies that the resulting 2-form is independent from the vertical coordinate t. □

From lemma 1 we deduce that definition 1 implies the reduction property stated in definition 3. To go in the other direction we just need to paste the local 2-forms $\Omega_{\mathbb{R}^{2n}}$ of lemma 1 to obtain a 2-calibration for $\mathcal{F}$.

The problem of deciding which manifolds admit a 2-calibrated foliation can be divided in several (very hard) subproblems: a 2-calibrated foliation $(M, \mathcal{F}, \omega)$ is the superposition of several compatible structures. Firstly the foliation structure. Secondly the 2-form restricts to a closed non-degenerate foliated 2-form $\omega_{\mathcal{F}}$; that defines a regular Poisson structure (M, Λ), $\Lambda \in \mathfrak{X}^2(M)$, $\Lambda_{\mathcal{F}}^{-1} = \omega_{\mathcal{F}}$, whose symplectic leaves are the leaves of $\mathcal{F}$. And thirdly, the foliated symplectic form is seen to admit a lift to a global closed 2-form ω (or a transversal direction along which the Poisson structure is invariant).

The second of the aforementioned steps, i.e. deciding which codimension one foliations are the symplectic foliations of a Poisson structure, is very complicated. The only partial result applies to open manifolds (see [3, 4]), where even the notion of openness is more involved than the usual one, for it depends not only on M but on the pair $(M, \mathcal{F})$ (it relies on the existence of certain Morse function compatible with the foliation).

Regarding the third step, the existence of the lift for the foliated 2-form $\omega_{\mathcal{F}}$ is obstructed in general: associated to a foliation $\mathcal{F}$ there are two natural cochain complexes. The first one is the subcomplex of the de Rham complex of *basic forms*, the forms vanishing along the directions of $\mathcal{F}$ and with vanishing Lie derivative w.r.t. (local) vector fields tangent to the foliation. This subcomplex is preserved by the exterior derivative and its cohomology is called basic cohomology.

The quotient of the de Rham complex by the subcomplex of basic forms is the complex of foliated forms, giving rise to the foliated cohomology groups $H^q(\mathcal{F})$. There is a natural spectral sequence relating both cohomologies (and whose $E_1^{0,q}$ groups compute the foliated cohomology groups).

The existence of a lift for $\omega_{\mathcal{F}}$ to a global closed 2-form follows from the vanishing of three obstructions associated to this spectral sequence (see for example [1]). In dimension 3 any co-oriented foliation by surfaces admits a leafwise area form, but according to a classical result of Sullivan only taut foliations posses leafwise area forms coming from closed 2-forms.

We would like to see a 2-calibrated foliation as a codimension one regular Poisson manifold with a lift for $\omega_{\mathcal{F}}$ to a closed global 2-form ω. We are not fully interested in the 2-form ω, as the following definition reflects.

Definition 4. *Let $(M_i, \mathcal{F}_i, w_i)$, $i = 1, 2$, be two 2-calibrated foliations. The two structures are said to be equivalent if there exists a diffeomorphism $\phi\colon M_1 \to M_2$ such that*

- *ϕ is a Poisson morphism or equivalence (i.e. it preserves the foliations together with the leafwise 2-forms).*
- *$[\phi^* w_2] = [w_1]$ and ϕ preserves the co-orientation.*

According to the previous definition the identity map

$$\mathrm{Id}\colon (M, \mathcal{F}, \omega) \to (M, \mathcal{F}, \omega + \beta)$$

is an equivalence of 2-calibrated foliations if and only if β is a basic 2-form with $\beta = d\alpha$.

Example 1. *Let $(M^3, \mathcal{F}, \omega)$ be a 3-dimensional taut foliation and (P, Ω) a symplectic manifold. Then the product $M \times P$ with foliation $\mathcal{F} \times M$ and 2-form $p_1^*\omega + p_2^*\Omega$ is a 2-calibrated foliation (p_1, p_2 are the projections onto each factor).*

Example 2. *Let (P, Ω) be a symplectic manifold and $\psi\colon P \to P$ a symplectomorphism. The mapping torus associated to ψ, $P \times [0,1]/(x, 1) \sim (\psi(x), 0)$, is a fiber bundle over S^1. The fibers foliate the manifold and the pullback of Ω to $P \times [0, 1]$ descends to the quotient to define a 2-calibration.*

If $\Psi, \Psi' \in \mathrm{Symp}(P, \Omega)$ are isotopic through symplectomorphisms, then if $\mathrm{dim} P \geq 4$ the resulting mapping torus are equivalent as 2-calibrated foliations (see the proof of the uniqueness statement of proposition 2). Conversely, if a mapping torus $M \to S^1$ carries two equivalent 2-calibrations Ω and Ω', then for any fixed leaf the corresponding return symplectomorphisms Ψ and Ψ' can be joined by a path of symplectomorphisms: the convex combination $(1 - t)\Omega + t\Omega'$ defines a path of 2-calibrations giving thus the desired path of symplectomorphisms.

Example 3. *In $\mathbb{R}^5$ consider the canonical 2-form $\omega_{\mathbb{R}^5}$. It descends to a closed 2-form $\omega_{\mathbb{T}^5}$ to $\mathbb{T}^5 = \mathbb{R}^5/\mathbb{Z}^5$. Let $\mathcal{F}$ be any of the foliations on $\mathbb{T}^5$ induced by a foliation of $\mathbb{R}^5$ by hyperplanes transversal to the vertical coordinate $t = x_5$, so that the hyperplane through the origin only intersects the integer lattice at the origin. Then $(\mathbb{T}^5, \mathcal{F}, \omega_{\mathbb{T}^5})$ is a 2-calibrated foliation with open leaves diffeomorphic to $\mathbb{R}^4$.*

Actually, since it is known that a co-oriented codimension one foliation is a mapping torus if and only if its leaves are compact, the second example describes all 2-calibrated foliations with compact leaves.

If $(M, \mathcal{F}, \omega)$ is a 2-calibrated foliation, then if $\mathcal{F}'$ is close enough to $\mathcal{F}$ (as distributions), then $(M, \mathcal{F}, \omega)$ is a 2-calibrated foliation. Example 3 can be constructed by slightly perturbing (the distribution of) a 2-calibrated foliation of $\mathbb{T}^5$ with compact leaves.

Lemma 2. *Let $\mathcal{F}$ be a co-oriented codimension one foliation on M closed and oriented manifold, and let $l\colon W \hookrightarrow M$ be a submanifold such that (i) l is transversal to $\mathcal{F}$ and hence W inherits a structure of foliated space $(W, \mathcal{F}_W)$, and (ii) each leaf of $\mathcal{F}$ intersects W in a unique connected component. Then we have:*

(1) *The embedding descends to a bijection $\tilde{l}\colon W/\mathcal{F}_W \to Z/\mathcal{F}$ which is a homeomorphism.*
(2) *If $\mathcal{F}_x$ is a leaf with polynomial growth, then $\mathcal{F}_{W,x} := \mathcal{F}_x \cap W$ has polynomial growth. Equivalently, if $\mathcal{F}_{W,x}$ has exponential growth then $\mathcal{F}_x$ has exponential growth.*
(3) *$\mathcal{F}_x$ is compact if and only if $\mathcal{F}_{W,x}$ is compact.*

Proof. The map is a bijection by condition (ii).

Open sets of $W/\mathcal{F}_W$ (resp. $M/\mathcal{F}$) are in one to one correspondence with saturated open sets of W (resp. M).

Let V be an open saturated set of $(M, \mathcal{F})$. By definition $V \cap W$ is an open set of W which is is clearly saturated (even without the assumption of $\tilde{l}$ being a bijection).

Now let V be an open saturated set of $(W, \mathcal{F}_W)$. We want to show that its saturation in $(M, \mathcal{F})$, denoted by $\bar{V}^{M,\mathcal{F}}$, is open and does not include any other point of W.

First of all recall that if V is a saturated set and $x \in V$, then x is an interior point if and only if for some T_x a local manifold through x transversal to the foliation, x is an interior point of $T_x \cap V$.

Hence, every $x \in V$ is an interior point of $\bar{V}^{M,\mathcal{F}}$. By using the holonomy, if a point in a leaf is interior the whole leaf is made of interior points. Since every leaf of $\bar{V}^{M,\mathcal{F}}$ intersects V, $\bar{V}^{M,\mathcal{F}}$ is open.

We now use that $\tilde{l}$ is a bijection to conclude that $V = V \cap \bar{V}^{M,\mathcal{F}}$, and this proves point 1. Notice that compactness of M is not required.

The two statements in point 2 are equivalent, because following Plante [33] the leaves in a compact co-oriented codimension one foliation have either exponential or polynomial growth. To prove them one can observe that any leaf $\mathcal{F}_x$ with polynomial growth is in the support of an invariant transversal measure; this measure restricts to an invariant transversal measure in $(W, \mathcal{F}_W)$ with $\mathcal{F}_{W,x}$ in its support.

It is clear that compact leaves in M give rise to compact leaves in W. Conversely, assume that $\mathcal{F}_x$ is not compact. We know that if y is a point in $\bar{\mathcal{F}}_x \backslash \mathcal{F}_x$, then $\mathcal{F}_y$ -the leaf through y- belongs to $\bar{\mathcal{F}}_x \backslash \mathcal{F}_x$. By hypothesis we have a point $z \in \mathcal{F}_y \cap W$. If we take a chart adapted to $\mathcal{F}$ and centered at z we will have a sequence of plaques of $\mathcal{F}_x$ accumulating in the plaque of $\mathcal{F}_y$ containing z. Since W is transversal to $\mathcal{F}$, it will have non-empty intersection with all plaques close enough to z. Thus, we

can construct a sequence of points $w_n \in \mathcal{F}_x \cap W$ whose limit z does not belong to $\mathcal{F}_x \cap W$. □

Theorem 1 will follow from the existence of a submanifold W^3 satisfying the hypothesis of lemma 2. In example 1 if we fix a point $p \in P$, then $M^3 \times \{p\}$ is a submanifold of $M \times P$ fulfilling the conditions of lemma 2. Since there is no restriction for the growth type of the leaves of a taut foliation, we conclude that the same happens for the leaves of 2-calibrated foliations.

We are interested in constructing as much examples as possible of (integral) 2-calibrated foliations. To do that we will introduce two surgery constructions.

2. Normal connected sum

The symplectic normal connected sum is a surgery construction in which two symplectic manifolds are glued along two copies of the same codimension 2 symplectic submanifold, which enters in the manifolds with opposite normal bundles (see [14]).

This surgery construction can be extended to regular Poisson manifolds, where the submanifold we glue along is of codimension two and inherits (i) the same Poisson structure with compact symplectic leaves from both embeddings and (ii) opposite normal bundles (see [20] for definitions and results). If the Poisson structures are induced from 2-calibrated foliations $(M_j, \mathcal{F}_j, \omega_j)$, $j = 1, 2$, then the Poisson normal connected sum $M_{1\#_\psi}M_2$ (along and appropriate submanifold and with gluing map ψ) is another regular Poisson manifold with codimension 1 leaves. As we mentioned in the previous section, the existence of a lift for the Poisson structure to a 2-calibrated structure can be studied through a spectral sequence. We are going to give an effective construction of the lift under some extra hypothesis.

Theorem 2. *Let $(M_j^{2n+1}, \mathcal{F}_j, \omega_j)$, $j = 1, 2$, be two integral 2-calibrated foliations. Let $(N^{2n-1}, \mathcal{F}_N, \omega_N)$ be a 2-calibrated foliation which is a mapping torus (the foliation has compact leaves). Assume that we have two maps $l_j\colon N \hookrightarrow M_j$ embedding N as a 2-calibrated submanifold of M_j (definition 2), and such that:*

1. *$H^2(N; \mathbb{Z})$ has no torsion.*
2. *The (compact) leaves of N have vanishing first (real) cohomology group.*
3. *The normal bundles of the embeddings are trivial.*
4. *The 2-calibrated foliations induced by the embeddings are equivalent to the original one $(N, \mathcal{F}_N, \omega_N)$ (definition 4).*

Then for a choice of gluing map ψ, there are Poisson structures Λ defined in the normal connected sum $M_{1\#_\psi}M_2$ that admit a lift to an integral closed 2-form ω.

Proof. Let Λ_j, $j = 1, 2$, denote the underlying Poisson structure on $(M_j, \mathcal{F}_j, \omega_j)$.

Let x, y be coordinates on $\mathbb{R}^2$ and r, θ the corresponding polar coordinates. Let $D(r)$ denote the open disk of radius r. Fix metrics on M_1 and M_2 and define the tubular neighborhoods

$$\mathcal{N}_{l_j(N)}(r) := \{x \in M_j \mid d(x, l_j(N)) \leq r\},\ j = 1, 2,$$

Since the normal bundle of $l_j(N)$ is trivial, for $\delta > 0$ small enough we have identifications

$$\mathcal{N}_{l_j(N)}(2\delta) \cong D(2\delta) \times l_j(N),\ j = 1, 2, \tag{2}$$

that can be chosen so that we have the following equality *along the leaves* [20]:

$$\omega_j = p_2^*\omega_{l_j(N)} + p_1^*(dx \wedge dy) \tag{3}$$

where p_1, p_2 denote the projections of $D(2\delta) \times l_j(N)$ onto the first and second factor respectively.

Let $A_j = (D(\delta)\backslash\{0\}) \times l_j(N) \subset M_j$, $j = 1,2$. The gluing map that defines $M_{1\#_\psi}M_2$ identifies A_1 with A_2 as follows:

$$\begin{array}{rcl} \psi\colon (D(\delta)\backslash\{0\}) \times l_1(N) & \longrightarrow & (D(\delta)\backslash\{0\}) \times l_2(N) \\ (r,\theta,q) & \longmapsto & (\sqrt{\delta^2 - r^2}, -\theta, l_2 \circ l_1^{-1}(q)) \end{array} \tag{4}$$

Equations 3 and 4 imply that the Poisson structures Λ_1 and Λ_2 are compatible with ψ, and therefore induce a Poisson structure Λ on $M_{1\#_\psi}M_2$ whose associated foliation we denote by $\mathcal{F}$.

We want to define the lift ω as i times the curvature of a hermitian complex line bundle with compatible connection.

Let us fix h_j an integral lift of $[\omega_j]$. Then we have a unique (isomorphism class of) hermitian complex line bundle with compatible connection (L_j, ∇_j) such that $c_1(L_j) = h_j/2\pi$ and $iF_j = \omega_j$.

The pullbacks $L_{N,1} := l_1^* L_1$ and $L_{N,2} := l_2^* L_2$ are isomorphic bundles because for both the curvatures $\omega_{N,j} := l_j^* \omega_j$, $j = 1,2$, define the same real cohomology class (condition 4), and since the integral cohomology has no torsion (condition 1) they are representatives of the unique isomorphism class of hermitian complex line bundle with Chern class $[\omega_{N,1}/2\pi] = [\omega_{N,2}/2\pi]$. Let us fix

$$\Psi_0 \colon L_{N,1} \to L_{N,2} \tag{5}$$

a (hermitian) bundle isomorphism.

We want to show that $\psi\colon A_1 \to A_2$ lifts to an isomorphism $\Psi\colon L_{1|A_1} \to L_{2|A_2}$ of complex hermitian line bundles.

Let $S_{j,r} \cong S^1 \times l_j(N)$, $r \in (0,\delta)$, denote the points in A_j with fixed radial coordinate r. We have $L_{j|S_{j,r}} \cong p_2^* L_{N,j|S_{j,r}}$. Hence Ψ_0 in equation 5 induces an identification

$$\Psi_r \colon L_{1|S_{1,r}} \to L_{2|S_{2,\sqrt{\delta^2-r^2}}}$$

Putting together all the bundle maps Ψ_r, $r \in (0,\delta)$, we get a bundle isomorphism

$$\Psi \colon L_{1|A_1} \to L_{2|A_2}$$

Notice that if $\Psi_0' \colon L_{N,1} \to L_{N,2}$ is isotopic to Ψ_0, then Ψ' and Ψ are also isotopic.

Using Ψ we obtain a hermitian line bundle $L_{1\#_\Psi}L_2 \to M_{1\#_\psi}M_2$. This line bundle has two not everywhere defined hermitian connections ∇_1, ∇_2, which overlap in $A_1 \subset M_{1\#_\psi}M_2$. We want to modify them so they can be glued to define a global connection whose leafwise curvature is $-i\Lambda^{-1}$.

Step 1: Modify ∇_1 so that the restriction of both connections to N coincides *along the leaves*.

From now on we omit the identifications l_1, l_2 and ψ. Let $\nabla_{N,j}$, $j = 1,2$, denote the restriction of ∇_j to $L_{N,j}$, and let $\alpha \in \Omega^1(N)$ be the difference $-i\nabla_{N,1} + i\Psi_0^*\nabla_{N,2}$. Since $d\alpha = -\omega_{N,1} + \omega_{N,2}$, by condition 4 α is leafwise closed. From condition 2 we deduce the existence on each (compact) leaf of N of a function whose derivative is α. It is also possible to make a choice on each leaf so that the resulting function $f\colon N \to \mathbb{R}$ is smooth.

Let $\beta\colon [0,2\delta] \to [0,1]$ be a standard cut-off function of a single variable with $\beta_{|[0,\delta]} = 1$, $\beta_{|[3\delta/2,2\delta]} = 0$. It induces a function $\beta \in C^\infty(M_1)$ by letting r be the radial coordinate in the fixed parametrization of $\mathcal{N}_N(2\delta)$ in equation 2 and by declaring it to vanish on the complement of the aforementioned neighborhood.

Consider the new hermitian connection on L_1

$$\bar{\nabla}_1 := \nabla_1 - id(\beta p_2^* f)$$

By construction $F_{\bar{\nabla}_1} = F_{\nabla_1}$ and the restriction of $\bar{\nabla}_1$ to $L_{N,1}$ equals $\Psi_0^*\nabla_{N,2}$ *along the leaves*.

Let us still call ∇_1 the new connection on L_1.

Step 2: Modify ∇_1, ∇_2 to have a leafwise normal form on $D(\delta) \times N$.

On $D(2\delta) \times N \subset M_j$, $j = 1, 2$, consider the connection

$$\nabla_j^\nu := p_2^*\nabla_{N,j} - i/2p_1^*(xdy - ydx) = p_2^*\nabla_{N,j} + i/2p_1^*(r^2 d\theta)$$

According to equation 3, along the leaves of $D(2\delta) \times N \subset M_j$ we have

$$F_{\nabla_j^\nu} = -i\omega_j$$

Hence $\eta_j := -i\nabla_j^\nu + i\nabla_j$ is a 1-form on $D(2\delta) \times N$ which is leafwise closed. Again, condition 2 implies the existence of smooth functions $h_j \colon D^2(2\delta) \times N \to \mathbb{R}$ such that $d_{\mathcal{F}_j} h_j = \eta_{j|\mathcal{F}_j}$.

Consider the new hermitian connections on L_j

$$\bar{\nabla}_j = \nabla_j - id(\beta h_j)$$

It can be checked that

(1) $F_{\bar{\nabla}_j} = F_{\nabla_j}$
(2) The restriction of $\bar{\nabla}_j$ to $L_{j|D(\delta)\times N}$ equals ∇_j^ν along the leaves

Let us still call ∇_1 and ∇_2 the connections matching ∇_1^ν and ∇_2^ν in $D(\delta) \times N$ respectively.

Step 3: Study the difference $\nabla_1^\nu - \Psi^*\nabla_2^\nu$ in $A_1 \subset M_1$ *along the leaves.*

By construction,

$$\nabla_1^\nu - \Psi^*\nabla_2^\nu = p_2^*\nabla_{N,1} + i/2p_1^*(r^2 d\theta) - (\Psi^*(p_2^*\nabla_{N,2} + i/2p_1^*(r^2 d\theta))) \tag{6}$$

Step 1 gives

$$p_2^*\nabla_{N,1} - \Psi^* p_2^*\nabla_{N,2} = 0$$

along the leaves, and hence equation 6 restricted to the leaves becomes

$$\nabla_1^\nu - \Psi^*\nabla_2^\nu = i/2p_1^*(r^2 d\theta) - (\Psi^* i/2p_1^*(r^2 d\theta)) \tag{7}$$

Now by equation 4

$$\Psi^* i/2p_1^*(r^2 d\theta) = i/2r^2 p_1^*(d\theta) - i/2\delta^2 p_1^* d\theta$$

and we get the leafwise equality

$$\nabla_1^\nu - \Psi^*\nabla_2^\nu = i/2\delta^2 p_1^* d\theta \tag{8}$$

Step 4: Make a final correction on ∇_2.

Let $\bar{\nabla}_2 = \nabla_2 - i/2\delta^2 d(\beta p_1^*\theta)$.

By the previous step, along the leaves of the "annular region" A_1 we have

$$\nabla_1 - \Psi^*\bar{\nabla}_2 = i/2\delta^2 p_1^* d\theta - i/2\delta^2 d(\varphi^*\beta p_1^*\theta), \tag{9}$$

and since on A_1 we have

$$d(\varphi^*\beta p_1^*\theta) = \beta p_1^* d\theta,$$

we conclude that equation 9 is a leafwise equality.

Let $\bar{\beta} \colon [0, 2\delta] \to [0, 1]$ be a cut-off function of one variable with $\bar{\beta}_{|[0,\delta/2]} = 1$, $\bar{\beta}_{|[\delta,2\delta]} = 0$. Let f denote also the corresponding cut-off function induced on $M_1 \backslash N$ by letting r be the radial coordinate in the fixed parametrization of $\mathcal{N}_N(2\delta)$.

We can define a global hermitian connection ∇ on $L_{1\#_\Psi}L_2 \to M_{1\#_\psi}M_2$ by the formula

$$\nabla = \begin{cases} (1-\bar{\beta})\nabla_1 + \bar{\beta}\Psi^*\bar{\nabla}_2 & \text{in } M_1\backslash N \subset M_{1\#_\psi}M_2 \\ \bar{\nabla}_2 & \text{in } M_2\backslash A_2 \subset M_{1\#_\psi}M_2 \end{cases} \tag{10}$$

The connection ∇ is well defined. Let $\omega := iF_\nabla$. Away from $A_1 \subset M_{1\#_\psi}M_2$ it is clear that $\omega_{\mathcal{F}} = \Lambda^{-1}$. Since the connections $\nabla_1, \bar{\nabla}_2$ coincide in the leaves of A_1, then $\omega_{\mathcal{F}} = \omega_{1|\mathcal{F}_1} = \Lambda^{-1}$ in that region also. Therefore ω is a lift for Λ^{-1}.

□

Proposition 1. *The equivalence class of the 2-calibrated structure defined in theorem 2 only depends on the isotopy classes of ψ and $\Psi_0\colon L_{N,1} \to L_{N,2}$.*

Proof. Let us fix ψ. By construction the Poisson structure associated to isotopic lifts Ψ_0 and Ψ_0' is the same (it does not depend on the bundle maps that lift ψ). Therefore, we only need to check that the cohomology class of the 2-calibration is the same (here the equivalence will be given by the identity map). This is equivalent to showing that the complex line bundles $L_{1\#_\Psi}L_2$ and $L_{1\#_{\Psi'}}L_2$ are isomorphic, and the isomorphism is easily constructed using a isotopy connecting the identity with $\Psi_0' \circ \Psi_0$.

If in the normal connected sum we choose isotopic identifications ψ and ψ' of the normal bundles of $l_1(N)$ and $l_2(N)$, then by the results of [20] there is a Poisson equivalence $\phi\colon (M_{1\#_\psi}M_2, \Lambda) \to (M_{1\#_{\psi'}}M_2, \Lambda')$.

It is easy to check that if Ψ_0 and Ψ_0' are isotopic then $L_{1\#_\Psi}L_2$ and $\phi^*L_{1\#_{\Psi'}}L_2$ are isomorphic, and thus ϕ is an equivalence of 2-calibrated foliations. □

The normal connected sum can be used to construct 2-calibrated foliations that use as building blocks the 2-calibrated foliations of examples 1 and 2, but which are neither products nor mapping tori.

Example 4. *Let (P^4, Ω) be a symplectic 4-manifold such that it contains a symplectic sphere S^2 with trivial normal bundle; let A be the induced area form on the sphere. Let $\varphi \in Symp(P, \Omega)$ such that $\varphi_{|S^2} = Id$ (for example φ can be the identity itself). We define $(M_1, \mathcal{F}_1, \omega_1)$ to be the mapping torus associated to φ.*

Let $(M_2, \mathcal{F}_2, \omega_2)$ be a 2-calibrated foliation as in example 2 using as factors any taut foliation $(Y^3, \mathcal{F}, \omega)$ with some non-compact leaves, and the sphere (S^2, A). Let C be a fixed transversal cycle for $(Y^3, \mathcal{F}, \omega)$ and $\theta\colon S^1 \to C$ any fixed positive parametrization (w.r.t. the co-orientation).

Let N^3 be result of applying the mapping torus construction to $\mathrm{Id} \in \mathrm{Symp}(S^2, A)$ *($N \cong S^1 \times S^2$). Since $\varphi_{|S^2} = Id$, there is an obvious embedding $l_1\colon N \hookrightarrow M_1$. The embedding l_2 is the product map $\theta \times Id\colon N \hookrightarrow M_2$.*

By construction the embeddings fulfill the hypothesis of theorem 2, so we obtain a 2-calibrated foliation $(M_{1\#_\psi}M_2, \mathcal{F}, \omega)$.

There is a one to one correspondence between the leaves of $(Y^3, \mathcal{F})$ and the leaves of $(M_{1\#_\psi}M_2, \mathcal{F})$. This correspondence sends compact leaves to compact leaves and non-compact leaves to non-compact leaves. Since $(Y^3, \mathcal{F})$ is assumed to have non-compact leaves $(M_{1\#_\psi}M_2, \mathcal{F}, \omega)$ is not a mapping torus.

If $(Y^3, \mathcal{F})$ is further assumed to have a compact leaf Σ and $C \cap \Sigma$ is a point, then $(M_{1\#_\psi}M_2, \mathcal{F}, \omega)$ has a compact leaf $\mathcal{F}_c$ which is the symplectic connected sum of $(\Sigma \times S^2, p_1^*\omega_{|\Sigma} + p_2^*A)$ and (P, Ω) along a trivial symplectic sphere. It is not hard to find symplectic 4-manifolds (P, Ω) so that $\mathcal{F}_c$ is not a product and therefore the resulting 2-calibrated foliation is not the product of a taut foliation with a surface. For example, we can take P to be the non-trivial S^2-bundle over S^2 with a symplectic

structure that makes a fiber a symplectic sphere. It can be checked that $\mathcal{F}_c$ is then a non-trivial S^2-bundle over Σ.

Remark 1. *In [20] it was shown that the normal connected sum could be used to construct 5-dimensional simply connected regular Poisson manifolds with codimension one leaves. Those methods, however, cannot be used to construct simply connected 2-calibrated foliations, since the pieces used in the process are Poisson but not 2-calibrated and the submanifolds do not have simply connected leaves (condition 2 in theorem 2).*

3. Generalized Dehn surgery

The second surgery we want to introduce is done, unlike the normal connected sum, along a submanifold inside one of the leaves.

Let $(M, \mathcal{F}, \omega)$ be a 2-calibrated foliation. We orient M so that a positive transversal vector (w.r.t. the co-orientation) followed by a positive basis of the leaf w.r.t. to the Liouville volume form associated to $\omega_{\mathcal{F}}$, gives a positive basis. Let us also assume that we have fixed a metric g on M.

In all what follows our reference concerning generalized Dehn twists and symplectic monodromy is the first section of [37], and our notation is mostly taken from there.

Let $T := T^*S^n$ and $T(\lambda)$ the subspace of cotangent vectors of length $\leq \lambda$, and

$$u\colon T \to \mathbb{R}$$

the length function. The cotangent bundle T carries a canonical symplectic structure $d\alpha_{\mathrm{can}}$, for which the zero section $T(0) \subset T$ is a lagrangian submanifold.

Using the round metric to identify T^*S^n with TS^n, the hamiltonian flow of $u^2/2$ is seen to be the normalized geodesic flow. For time π it can be extended over $T(0)$ to a diffeomorphism

$$\sigma\colon T \to T$$

The restriction of σ to the zero section $T(0)$ is the antipodal map.

For any fixed $\lambda > 0$, the time 2π hamiltonian flow of $R(u)$, with $R\colon \mathbb{R} \to \mathbb{R}$ a suitable function, gives rise to symplectomorphisms

$$\tau\colon T \to T$$

supported in the interior $T(\lambda)$ (i.e. they are the identity near the boundary of $T(\lambda)$). Any of them is called a *model generalized Dehn twist*. All them are isotopic through an isotopy in $\mathrm{Symp}^{\mathrm{comp}}(T(\lambda))$, the group of symplectomorphisms of $(T(\lambda), d\alpha_{\mathrm{can}})$ supported in the interior of $T(\lambda)$.

Let L be a parametrized lagrangian sphere, i.e. $l\colon S^n \hookrightarrow (M, \mathcal{F}, \omega)$, with $L = l(S^n)$ contained in one leaf and $\omega_{|L} \equiv 0$.

Let $\mathcal{F}_L$ denote the leaf containing L. There exists U a neighborhood of L in $\mathcal{F}_L$ and $\lambda > 0$ such that we can find an extension of $l^{-1}\colon L \to S^n$ to a symplectomorphism

$$\varphi\colon (U, \omega_{\mathcal{F}}) \to (T(\lambda), d\alpha_{\mathrm{can}}) \tag{11}$$

Let us cut $(M, \mathcal{F}, \omega)$ open along U, so that we obtain two copies of it. According to the co-orientation, there is a negative one U^- and a positive one U^+ (the flow of a positive transversal vector field goes from the negative to the positive).

Let L be a parametrized lagrangian sphere. If $n = 1$ assume that the L is a loop with trivial holonomy (this is granted for $n > 1$ by Reeb's stability theorem).

Definition 5. *The generalized Dehn surgery of $(M, \mathcal{F}, \omega)$ along L is defined by cutting M open along U and then gluing back via the composition*

$$\chi\colon (U^-, \omega_{\mathcal{F}}) \xrightarrow{\varphi} (T(\lambda), d\alpha_{\mathrm{can}}) \xrightarrow{\tau} (T(\lambda), d\alpha_{\mathrm{can}}) \xrightarrow{\varphi^{-1}} (U^+, \omega_{\mathcal{F}}), \tag{12}$$

where τ is any choice of model generalized Dehn twist supported in $T(\lambda)$.

We denote the corresponding manifold by M^L.

Proposition 2. *M^L, which carries and obvious foliation $\mathcal{F}^L$, admits a 2-calibration ω^L which in principle depends on the pair (φ, τ). If $n > 1$ then*

(1) *ω^L is unique up to equivalence.*
(2) *$[\omega]$ is integral if and only if $[\omega^L]$ is integral.*
(3) *$\pi_j(M^L) \cong \pi_j(M)$ and $H_j(M^L; \mathbb{Z}) \cong H_j(M; \mathbb{Z})$, $0 \leq j \leq n-2$.*

Proof. Let R be a positive vector field in the kernel of ω defined in a neighborhood of U, and whose flow φ_t^R preserves $\mathcal{F}$. Let $\epsilon > 0$ small enough so that

$$\begin{aligned} \varphi^R\colon [-\epsilon, \epsilon] \times U &\longrightarrow M \\ (t, x) &\longmapsto \varphi_t^R(x) \end{aligned} \tag{13}$$

is an embedding. We introduce the following notation:

$$\begin{aligned} U(\epsilon) &:= \varphi^R([-\epsilon, \epsilon] \times U), & U_t &:= \varphi_t^R(U) \\ U^+(\epsilon) &:= \varphi^R([0, \epsilon] \times U), & U^-(\epsilon) &:= \varphi^R([-\epsilon, 0] \times U) \end{aligned} \tag{14}$$

After cutting open $U(\epsilon)$ along U we get two sets $U^-(\epsilon), U^+(\epsilon)$. Once we glue using the identification χ of equation 12 we obtain $U^L(\epsilon) := U^-(\epsilon) \bigcup_\chi U^+(\epsilon) \subset M^L$, which carries a vector field R^L transversal to the induced foliation $\mathcal{F}^L$ on M^L (and preserving it).

Since R preserves ω and the foliation, the restriction of ω to $U^-(\epsilon)$ and $U^+(\epsilon)$ defines closed 2-forms ω^- and ω^+ independent of the time coordinate. When we glue U^- to U^+ using χ, being this map a symplectomorphism, the forms ω^- and ω^+ descend to $U^L(\epsilon)$ to a 2-form ω_ϵ^L which is clearly closed. Then

$$\omega^L := \begin{cases} \omega & \text{in } M^L \backslash U^L(\epsilon) \\ \omega_\epsilon^L & \text{in } U^L(\epsilon) \end{cases}$$

is a 2-calibration for $\mathcal{F}^L$.

To prove the uniqueness, let $\tilde{\tau}$ be another model generalized Dehn twist and $(\tilde{M}^L, \tilde{\mathcal{F}}^L, \tilde{\omega}^L)$ the 2-calibrated foliation constructed as above, but using $\tilde{\tau}$ instead of τ in equation 12. By hypothesis, it gives rise to a identification

$$\tilde{\chi}\colon (U^-, \omega_{\mathcal{F}}) \to (U^+, \omega_{\mathcal{F}})$$

which is isotopic to χ. Hence we have

$$\Psi\colon [0,1] \times (U, \omega_{\mathcal{F}}) \to (U, \omega_{\mathcal{F}})$$

such that

- Ψ_t, $t \in [0,1]$, is a compactly supported symplectomorphism.
- $\Psi_0 = \mathrm{Id}$ and $\Psi_1 = \tilde{\chi} \circ \chi^{-1}$.

Let $\beta\colon [0,1] \to [0,1]$ be a cut-off function of one variable with $\beta_{|[0,1/3]} = 0$ and $\beta_{|[2/3,1]} = 1$.

To define $\phi\colon (M^L, \mathcal{F}^L, \omega_L) \to (\tilde{M}^L, \tilde{\mathcal{F}}^L, \tilde{\omega}^L)$ we identify $U_t \subset U^+(\epsilon)$, $t \in [0, \epsilon]$, with its image in M^L and $\tilde{M}^L$ and set

$$\phi = \begin{cases} \mathrm{Id} & \text{in } \ M^L \setminus \bigcup_{t\in[0,\epsilon]} U_t \\ \Psi_{\beta(1-t/\epsilon)} & \text{in } \ U_t, \ t\in[0,\epsilon] \end{cases}$$

The map ϕ is a well defined diffeomorphism. It is a Poisson equivalence because it preserves the foliation and restricts to each leaf to a symplectomorphism.

When $n>1$ the equality $[\phi^*\tilde{\omega}^L]=[\omega^L]$ follows from general position arguments: let $\delta>0$ such that $\nu(L)\cong\mathcal{N}_L(2\delta)$. We can assume that the neighborhood $U(\epsilon)$ used in the construction of M^L (and $\tilde{M}^L$) is contained in $\mathcal{N}_L(\delta)$. The images of $\mathcal{N}_L(\delta)$ in M^L and $\tilde{M}^L$ are tubular neighborhoods of $L\subset M^L$ and $L\subset\tilde{M}^L$ respectively, which contract onto L. The cohomology classes $[\phi^*\tilde{\omega}^L]$ and $[\omega^L]$ can be evaluated over embedded surfaces, because $\dim M\geq 5$. These surfaces can be isotoped to avoid $L\subset M^L$ and $L\subset\tilde{M}^L$ respectively, and thus they can be arranged to avoid the image of $\mathcal{N}_L(\delta)$ in M^L and $\tilde{M}^L$ respectively. Therefore the surfaces will be contained in the region where $\phi^*\tilde{\omega}^L=\omega=\omega^L$. As a consequence, $[\omega^L]$ is integral if and only if $[\omega]$ is so.

Similar arguments show that the equivalence class of the 2-calibration does not depend either on the identification φ in equation 11, because two such choices are isotopic by an isotopy supported in a compact neighborhood of the zero section, and this proves points 1 and 2.

The equalities of homology and homotopy groups also follow from general position arguments.

When $n=1$ we conclude that the Poisson structure on the resulting taut foliation is unique. □

Remark 2. *Recall that a "framed" lagrangian n-sphere* [36] *is a parametrized n-sphere up to isotopy and the action of O(n+1). Model generalized Dehn twists associated to two parametrizations defining the same "framed" lagrangian n-sphere are isotopic, the isotopy by symplectomorphisms supported in a compact neighborhood of the lagrangian sphere (remark 5.1 in* [36]*). Therefore the generalized Dehn surgery is well defined for "framed" lagrangian spheres.*

Remark 3. *We can use the flow of R to displace the lagrangian sphere L to a new lagrangian sphere L' in a nearby leaf. It follows that $(M^L,\mathcal{F}^L,\omega^L)$ and $(M^{L'},\mathcal{F}^{L'},\omega^{L'})$ are equivalent.*

Remark 4. *If we use instead of τ its inverse, we get a new 2-calibrated foliation $(M^{L^-},\mathcal{F}^{L^-},\omega^{L^-})$ referred to as negative generalized Dehn surgery on L; observe that negative generalized Dehn surgery is generalized Dehn surgery for the opposite co-orientation. It follows from remark 4 that generalized Dehn surgery along L followed by negative generalized Dehn surgery along L, yields the original 2-calibrated foliation. The composition in the oposite order is also the identity, so they are inverse of each other.*

3.1. **Lagrangian surgery.** It is possible to give a rather simple description of a (singular) foliated cobordism $(Z,\mathcal{W})$ from $(M,\mathcal{F})$ to $(M^L,\mathcal{F})$. Moreover, the cobordism can be shown to be the attaching of a symplectic (n+1)-handle to the trivial cobordism $M\times[-\varepsilon,\varepsilon]$. The boundary component M^L inherits a canonical structure of 2-calibrated foliation equivalent to the one coming from proposition 2.

To show that we start by recalling the following result:

Lemma 3. *The normal bundle $\nu(L)$ of the parametrized lagrangian sphere L is trivial and carries a canonical framing μ_L.*

Proof. The normal bundle inside the leaf is isomorphic to T^*L. The full normal bundle $\nu(L)$ is isomorphic to $\underline{\mathbb{R}}\oplus T^*L$. Using the round metric in the sphere to

identify cotangent and tangent bundle we have

$$\nu(L) \cong \underline{\mathbb{R}} \oplus TS^n \cong \underline{\mathbb{R}}^{n+1}{}_{|S^n},$$

where in the last isomorphism a positive generator of $\underline{\mathbb{R}}$ is sent to the outward normal unit vector field.

□

If we use the negative co-orientation we get a different framing $-\mu_L$. Let

$$r_{\mathrm{norm}}, r_{e_1} : \underline{\mathbb{R}}^{n+1}{}_{|S^n} \to \underline{\mathbb{R}}^{n+1}{}_{|S^n}$$

be the bundle maps that over each point of the sphere reflect the fiber along the hyperplane normal to the outward normal vector field and the vector field e_1 respectively (here $\mathbb{R}^{n+1}$ is trivialized by the basis $e_1, \ldots, e_{n+1}$ associated to some fixed coordinates). Then

$$-\mu_L = \mu_L \circ r_{\mathrm{norm}}$$

Observe that for a fixed orientation of the piece of (n+1)-handle to be attached, surgeries with framing μ_L and $-\mu_L$ respectively yield oriented manifolds if we start form opposite orientations of M. To get an oriented manifolds starting from a fixed orientation of M, we have to reverse the orientation of say the handle we attacht when using the framing $-\mu_L$; this is equivalent to considering instead the framing $\mu_{L^-} := -\mu_L \circ r_{e_1}$. Let $\Delta := r_{\mathrm{norm}} \circ r_{e_1}$, which is (up to sign) the image of $\pi_n(S^n)$ in $\pi_n(\mathrm{SO}(n+1))$ coming naturally from the long exact homotopy sequence of $(\mathrm{SO}(n+1), \mathrm{SO}(n))$.

By construction

$$\mu_{L^-} = \mu_L \circ \Delta \tag{15}$$

Remark 5. *Let L be a loop with trivial holonomy inside a leaf of a co-oriented 3-dimensional taut foliation $(M^3, \mathcal{F})$. Let R be a positively oriented vector field as in proposition 2. Then the push out of L along the flow of R defines a canonical framing μ. The framing μ_L coming from lemma 3 is $\mu - m$, where m is the meridian.*

In dimension 3 it is a classical result of Lickorish [26], that surgery on L with framing μ_L ((-1)-surgery) is diffeomorphic to M^L (a 2-dimensional model Dehn twist is a classical right-handed or positive Dehn twist). Gluing using a negative Dehn twist is equivalent to using the framing μ_{L^-} ((+1)-surgery). In dimension 2 the bundle map Δ amounts to adding two meridians to the longitude, and equation 15 says of course that (+1)-framing is obtained by twisting twice in the positive direction and then composing with the (-1)-framing.

Definition 6. *Let $L \subset M$ be a parametrized lagrangian sphere. Let Z be the cobordism which amounts to attaching to the trivial cobordism an (n+1)-handle along L with the canonical framing μ_L (resp. μ_{L^-}). We refer to the component of ∂Z different from M as the result of performing lagrangian surgery along L with framing μ_L (or just lagrangian surgery along L), and we denote it by M^{μ_L} (resp. $M^{\mu_{L^-}}$).*

Lickorish' result is known to extend to higher dimensions, i.e. the manifolds M^L (resp. M^{L^-}) and M^{μ_L} (resp. $M^{\mu_{L^-}}$) are diffeomorphic (see remark 10).

3.2. **Cosymplectic structures and symplectic (n+1)-handles.** In M^{μ_L} it is also possible to define a canonical 2-calibrated structure without using the aforementioned identification with M^L and proposition 2. The strategy is the same used in contact geometry to show that surgeries on legendrian spheres give rise to new contact manifolds [41].

Definition 7. *A cosymplectic structure on M^{2n+1} is a pair $\alpha \in \Omega^1(M)$, $\omega \in \Omega^2(M)$, $d\alpha = d\omega = 0$, and such that $\alpha \wedge \omega^n$ is a volume form.*

The Reeb vector field is the unique vector field R satisfying $i_R\alpha = 1$, $i_R\omega = 0$.

Example 5. *Let (P, Ω) be a symplectic manifold. Let $H \subset P$ be a hypersurface and Y a symplectic vector field transversal to H ($\mathcal{L}_Y\Omega = 0$). The hypersurface inherits a cosymplectic structure (H, α, ω), where α is the closed 1-form $i_Y\Omega_{|H}$ and $\omega := \Omega_{|H}$: the annihilator of Y w.r.t. Ω integrates into a codimension one foliation $\mathcal{D}$ of P, which is by construction transversal to H. The foliation* Kerα *is $\mathcal{D} \cap H$. The leaves are symplectic because for each $x \in H$, any subspace in $\mathrm{Ann}(Y)_x^{\Omega}$ supplementary to Y is symplectic.*

The vector field Y is defined in a neighborhood of H, and its flow can be used as in equation 13 to give a product structure $H \times [-\varepsilon, \varepsilon]$.

The construction of example 5 can be reversed:

Lemma 4. *The cosymplectic structure (H, α, ω) together with the product structure $H \times [-\varepsilon, \varepsilon]$ determine completely the symplectic structure on $H \times [-\varepsilon, \varepsilon]$. Changing the co-orientation (i.e. taking $-\alpha$) gives the opposite symplectic vector field.*

Proof. Let t be the coordinate of the interval. It is a simple calculation that

$$\Omega = \omega + d(t\alpha)$$

□

A consequence of lemma 4 is the following result.

Proposition 3. *Let $(P_0, \Omega_0), (P_1, \Omega_1)$ be symplectic manifolds. Let H_0 and H_1 be hypersurfaces and Y_0, Y_1 symplectic vector fields transversal to them (defined in tubular neighborhoods), so that we have product structures $H_0 \times [-\varepsilon, \varepsilon]$, $H_1 \times [-\varepsilon, \varepsilon]$. Let $(H_0, \mathcal{F}_0, \omega_0, \alpha_0)$, $(H_1, \mathcal{F}_1, \omega_1, \alpha_1)$ be the induced cosymplectic structures as described in example 5. Suppose that $\phi\colon H_0 \to H_1$ is a diffeomorphism such that $\phi^*\mathcal{F}_1 = \mathcal{F}_0$, $\phi^*\omega_1 = \omega_0$, $\phi^*\alpha_1 = \alpha_0$ (an equivalence of cosymplectic structures). Then*

$$\phi \times Id\colon (H_0 \times [-\varepsilon, \varepsilon], \Omega_0) \to (H_1 \times [-\varepsilon, \varepsilon], \Omega_1)$$

is a symplectomorphism.

Let (P_0, Ω_0), (P_1, Ω_1) be symplectic manifolds. Let $H_i \subset P_i$ be hypersurfaces and let Y_i be symplectic vector fields transversal to the hypersurfaces. Let $L_i \subset H_i$ be parametrizated lagrangian spheres contained in a leaf of the induced cosymplectic structures. Fix symplectomorphisms of small tubular neighborhoods of the spheres in their corresponding leaves $\varphi_i\colon (U_i, \Omega_{i|U_i}) \to (T(\lambda), d\alpha_{\mathrm{can}})$, for some $\lambda > 0$, extending the given parametrizations of L_i. Assuming $n > 1$ or L_i a loop without holonomy if $n = 1$, we use the Reeb vector fields to canonically extend the symplectomorphisms to a (local) equivalence of cosymplectic structures. Proposition 3 gives a symplectomorphism

$$\varpi\colon (V_0, H_0, L_0, \Omega_0) \to (V_1, H_1, L_1, \Omega_1) \tag{16}$$

The hypersurface H_i splits P_i, $i = 0, 1$, into two manifolds with boundary P_i^- and P_i^+, which contain the translates of H_i by the flow of Y_i for negative and positive values respectively.

Definition 8. *The symplectic connected sum of $(P_0, \Omega_0, H_0, Y_0)$ and $(P_1, \Omega_1, H_1, Y_1)$ along the parametrized lagrangian spheres L_0 and L_1 is the result of gluing P_0^- with P_1^+ along their boundaries using the collars described in example 5. The existence of the symplectomorphism ϖ implies that away from the points of $\partial(V_0 \cap H_0) \subset H_0$ (the corners), it carries a symplectic structure which only depends on the symplectomorphisms $\varphi_i\colon (U_i, \Omega_{i|U_i}) \to (T(\lambda), d\alpha_{\mathrm{can}})$.*

Proposition 4. *Let $(M, \mathcal{F}, \omega)$ be a 2-calibrated foliation and L a parametrized lagrangian sphere. Let Z be the cobordism which is the result of attaching to the trivial cobordism an (n+1)-handle along L with canonical framing μ_L, and let M^{μ_L} the boundary component different from M. Then the cobordism admits symplectic structures that induce 2-calibrated structures on M^{μ_L} (and the original one on M).*

Proof. We will realize the cobordism as the symplectic connected sum of two symplectic manifolds along parametrized lagrangian spheres.

Step 1: Fix the appropriate structures in the trivial cobordism $M \times [-\varepsilon, \varepsilon]$, or rather in the region where the (n+1)-handle will be attached.

Let $\nu(L) \cong \mathcal{N}_L(\delta)$, for some $\delta > 0$, be a small tubular neighborhood of L. We first lift the 2-calibrated foliation $(\nu(L), \mathcal{F}_{|\nu(L)}, \omega_{|\nu(L)})$ to a cosymplectic structure: let R be a vector field defined in $\nu(L)$ which belongs to Kerω, with positive orientation, and whose flow preserves $\mathcal{F}$. Let α be the 1-form defined by the conditions Ker$\alpha = \mathcal{F}$, $\alpha(R) = 1$. The triple $(\nu(L), \alpha, \omega_{|\nu(L)})$ is a cosymplectic structure -with Reeb vector field R- which lifts the original 2-calibrated foliation.

We consider the tuple

$$(P_0, \Omega_0, H_0, Y_0) := (\nu(L) \times [-\varepsilon, \varepsilon], \omega + d(t\alpha), \nu(L) \times \{0\}, \partial/\partial t) \tag{17}$$

The vector field Y_0 is symplectic in $\nu(L) \times [-\varepsilon, \varepsilon]$ (it is actually hamiltonian) and the induced cosymplectic structure on $\nu(L) \times \{0\}$ is the same already defined two paragraphs above (this is lemma 4).

Fix any symplectomorphism $\varphi\colon (U, \omega_{\mathcal{F}}) \to (T(\lambda), d\alpha_{\text{can}})$, $\lambda > 0$, and U a neighborhood of $L \times \{0\}$ in H_0, extending the given parametrization of L. This completes the needed data for the first summand.

Step 2: Choose symplectic model for the (n+1)-handle, hypersurface, symplectic vector field, lagrangian sphere and identification with $(T(\lambda), d\alpha_{\text{can}})$.

Consider the complex Morse function

$$\begin{array}{rcl} h\colon \mathbb{C}^{n+1} & \longrightarrow & \mathbb{C} \\ (z_1, \dots, z_{n+1}) & \longmapsto & z_1^2 + \dots + z_{n+1}^2 \end{array}$$

Let $\Omega_{\mathbb{C}^{n+1}} = d\alpha_{\mathbb{C}^{n+1}}$ be the standard symplectic form in $\mathbb{C}^{n+1}$ ($\Omega_{\mathbb{C}^{n+1}} = \Omega_{\mathbb{R}^{2n+2}}$). The fibers of h are symplectic submanifolds. Let $T^v h$ be the distribution of their tangent spaces (away from the critical point). Their symplectic orthogonals define a distribution by symplectic planes (a symplectic connection) away from the origin, the critical point of h. For any path $\gamma\colon [a, b] \to \mathbb{C}^*$ it defines parallel transport maps $\rho_\gamma\colon h^{-1}(\gamma(a)) \to h^{-1}(\gamma(b))$ which are symplectomorphisms. For each $z \in \mathbb{C}^*$, the sphere

$$\Sigma_z = \{(\sqrt{z} x_1, \dots, \sqrt{z} x_{n+1}) \mid x \in S^n \subset \mathbb{R}^{n+1}\} \subset h^{-1}(z) \tag{18}$$

is characterized as the subset of points of the fiber over z sent to the critical point by the parallel transport over the segment joining z with the origin; this is a lagrangian sphere. Denote by Σ the union of all these spheres and the origin.

In lemma 1.10 [37], Seidel describes a parametrization

$$\Phi\colon \mathbb{C}^{n+1} \backslash \Sigma \to \mathbb{C} \times (T \backslash T(0)) \tag{19}$$

which preserves the (exact) symplectic structures of the fibers; the map also parametrizes the whole fiber over the points $(r, 0)$, $r > 0$.

Let $\bar{D}(r) \subset \mathbb{R}^2 = \mathbb{C}$ be the closed disk of radius r, and r, θ polar coordinates in the plane. For each $r > 0$, the counter-clockwise parallel transport over $\partial \bar{D}(r)$ defines a symplectomorphism

$$\rho_{\bar{D}(r)}\colon h^{-1}(r) \to h^{-1}(r)$$

Its conjugation by the parametrization

$$\varphi_r := \Phi_{|h^{-1}(r)} \colon h^{-1}(r) \to T \tag{20}$$

defines a symplectomorphism

$$\tilde{\tau}_r \colon T \to T$$

which is the time 2π map of the hamiltonian of $\tilde{R}_r(u)$,

$$\tilde{R}_r(t) = \frac{1}{2}t^2 - \frac{1}{2}(t^2 + r^2/2)^{1/2} \tag{21}$$

Fix any $\lambda > 0$ and W a small enough neighborhood of the origin in $\mathbb{C}^{n+1}$. According to [37], for any appropriate cut-off function $g \in C^\infty(\mathbb{R}, \mathbb{R}^+)$ it is possible to find $\zeta \in \Omega^1(\mathbb{C}^{n+1})$ such that

- ζ vanishes in W.
- $\Omega_{\mathbb{C}^{n+1}} + d\zeta$ is symplectic.
- The restriction of $\Omega_{\mathbb{C}^{n+1}}$ and $\Omega_{\mathbb{C}^{n+1}} + d\zeta$ to the fibers of h coincide.
- For any $r > 0$ the conjugation of $\rho_{\bar{D}(r)}$ by φ_r defines a symplectomorphism
$$\tau_r \colon T \to T, \tag{22}$$
which is the time 2π of the hamiltonian flow of $g(u)\tilde{R}(u)$. Therefore it is a model Dehn twist supported in $T(\lambda)$.

For each $r, \lambda > 0$ the (n+1)-handle P_1 will be a contractible subset of $\mathbb{R}^{2n+2}$ -to be defined- whose image by h contains $\bar{D}(r)$. The symplectic form is the restriction of

$$\Omega_{\mathbb{C}^{n+1}} + d\zeta \in \Omega^2(\mathbb{C}^{n+1}) \tag{23}$$

Fix $\epsilon, r_0 > 0$ and for each $r \in [-r_0, 0) \cup (0, r_0]$ let $v_r(-\epsilon, \epsilon)$ be the vertical segment joining $(r, -\epsilon)$ and (r, ϵ).

For $r \in (0, r_0]$ let $H_{1,r}$ (resp. $H_r^{\mu_L}$) be $h^{-1}(v_r(-\epsilon, \epsilon))$ (resp. $h^{-1}(v_{-r}(-\epsilon, \epsilon))$), a piece of which is going to be our hypersurface (resp. part of the new boundary of the cobordism).

The 1-form $\alpha_{\mathbb{C}^{n+1}} + \zeta$ descends to each hypersurface $H_{1,r}$ (resp. $H_r^{\mu_L}$), $r \in (0, r_0]$, to a 1-form α_r, inducing thus an exact symplectic structure. We shall denote by $\alpha_{\mathcal{F},r}$ the foliated 1-form, and also by α_r the restriction of the 1-form to each of the symplectic leaves if there is no risk of confusion (sometimes we will add another subindex t parametrizing the spaces of leaves).

Let $\mathrm{Im}h \colon \mathbb{R}^{2n} \to \mathbb{R}$ be the imaginary part of h. We define Y_1 to be the hamiltonian vector field in $(\mathbb{R}^{2n+2}, \Omega_1)$ of $-\mathrm{Im}h$. By construction, Y_1 belongs to the symplectic annihilator w.r.t Ω_1 of $T^v h$, and $h_* Y_1(p)$ is an strictly negative multiple of $\partial/\partial x$; in fact Y_1 is the gradient vector field w.r.t. $g_0 = \Omega_{\mathbb{C}^{n+1}}(\cdot, J_0 \cdot)$ of $-\mathrm{Re}h$ (see lemma 1.13 in [37]). Since the horizontal lines are transversal to $v_r(-\epsilon, \epsilon)$, Y_1 is transversal to $H_{1,r}$ (resp. $H_r^{\mu_L}$). Therefore $H_{1,r}$ (resp. $H_r^{\mu_L}$) inherits a cosymplectic structure. Notice that the leaves of $\mathrm{Ann}(Y_1)^{\Omega_1}$ are h^{-1} of the horizontal lines. Thus the leaves of the induced cosymplectic structure on $H_{1,r}$ (resp. $H_r^{\mu_L}$) are h^{-1} of the points of $v_r(-\epsilon, \epsilon)$ (resp. $v_{-r}(-\epsilon, \epsilon)$).

The parametrized lagrangian sphere in $H_{1,r}$ is $\Sigma_r = \sqrt{r} S^n \subset \mathbb{R}^{n+1}$, and the symplectic identification of a neighborhood of the sphere in its leaf with $(T(\lambda), d\alpha_{\mathrm{can}})$ is φ_r in equation 20.

Step 3: Select an (n+1)-handle with the "appropriate shape".

We want the (n+1)-handle to be an appropriate subset $P_{1,r} \subset (\mathbb{R}^{2n+2}, \Omega_{\mathbb{C}^{n+1}} + d\zeta)$ so that

- the symplectic connected sum along $L \subset H_0$ and $\Sigma_r \subset H_{1,r}$, corresponds to the handle attaching along the parametrized lagragian sphere L with framing μ_L and,
- the boundary component of the cobordism $M_r^{\mu_L}$ inherits the right 2-calibrated foliation.

There is an induced local Reeb vector field R_r on $H_{1,r}$ (resp. $H_r^{\mu_L}$). Let R'_r be the multiple of R_r such that $h_* R'_r = \partial/\partial y$, $r \in [-r_0, 0) \cup (0, r_0]$. Notice that R'_r is a *negative transversal vector field.*

Consider the subsets

$$T_r(\lambda) := \begin{cases} \varphi_r^{-1}(T(\lambda)) & 0 < r \leq r_0 \\ \varphi_r^{-1}(T(\lambda) \backslash T(0)) \cup \Sigma_r & -r_0 \geq r \geq 0 \end{cases}$$

Define $T_r(\lambda, \epsilon)$ as in equation 14, where the flow is now given by R'_r in $H_{1,r}$ and in $H_r^{\mu_L}$.

For $r > 0$ let

$$f_r \in C^\infty(T_r(\lambda, \epsilon) \backslash \Sigma_r, \mathbb{R}^+) \tag{24}$$

with the following properties:

- The support of f_r is contained in the interior of $T_r(\lambda, \epsilon)$.
- $\varphi_1^{f_r Y_1}$ sends $T_r(5\lambda/6, 5\epsilon/6) \backslash \Sigma_r$ into $H_r^{\mu_L}$.

Consider the smooth surface

$$H_r^{\mu_L, 0} := \varphi_1^{f_r Y_1}(T_r(\lambda, \epsilon) \backslash \Sigma_r) \cup \Sigma_{-r} \tag{25}$$

We define $P_{1,r}$ to be the connected component of $\mathbb{C}^{n+1}$ bounded by $H_r^{\mu_L, 0}$ and $T_r(\lambda, \epsilon)$, and containing the origin.

Observe that $P_{1,r}$ is an (n+1)-handle (it is a thickening of $\cup_{r \in [-r_0, r_0]} \Sigma_r \cup \{0\}$, the union of the critical point and stable and unstable manifolds of Reh). By construction the symplectic connected sum is -from the point of view of differential topology- just attaching an (n+1)-handle to L with certain framing. The framing is the differential at $\Sigma_r \subset H_{1,r}$ of $\phi \colon (H_{1,r}, \Sigma_r) \to (\nu(L), L)$, which is seen to be isotopic to μ_L. Thus $M_r^{\mu_L} := H_r^{\mu_L, 0} \cup M \backslash T_r(\lambda, \epsilon)$, the component different from M of ∂Z, is diffeomorphic to M^L.

The hypersurface $H_r^{\mu_L, 0}$ is transversal to Y_1 and inherits a 2-calibrated foliation $(\mathcal{F}_r^{\mu_L}, \omega_r^{\mu_L})$, which is compatible with the induced 2-calibrated foliation on $(M \backslash supp(f_r)) \subset M$. Hence we get $(M_r^{\mu_L}, \mathcal{F}_r^{\mu_L}, \omega_r^{\mu_L})$, with $(M_r^{\mu_L}, \mathcal{F}_r^{\mu_L}) \cong (M^L, \mathcal{F}^L)$. $\square$

Remark 6. *Observe that instead of gluing the (n+1)-handle to the trivial cobordism, we can proceed the other way around. This amounts to reversing the coorientation on $(M, \mathcal{F}, \omega)$, and hence considering in the (n+1)-handle the opposite symplectic vector field* $\mathrm{Im}(h)$. *Actually, we can do things in an equivalent way: in the (2n+2)-dimensional (n+1)-handle we can use as attaching sphere Σ_{-r} instead of Σ_r, $r > 0$ (and also choosing an appropriate shape for the handle). We go from the second point of view to the first by using the symplectic transformation $(z_1, \ldots, z_{n+1}) \mapsto (-iz_1, \ldots, -iz_{n+1})$. One checks that the new boundary is a 2-calibrated foliation $(M_r^{-\mu_L}, \mathcal{F}_r^{-\mu_L}, \omega_r^{-\mu_L})$ diffeomorphic to $(-M^{L^-}, \mathcal{F}^{L^-})$ as cooriented foliations. If $(M_r^{\mu_{L^-}}, \mathcal{F}_r^{\mu_{L^-}}, \omega_r^{\mu_{L^-}})$ denotes $(M_r^{-\mu_L}, \mathcal{F}_r^{-\mu_L}, \omega_r^{-\mu_L})$ with the orientation reversed, then $M_r^{\mu_{L^-}}$ is obtained from M by surgery along L with framing μ_{L^-}.*

3.3. **Poisson morphisms and extensions of exact Poisson isotopies near submanifolds with trivial first real cohomology group.** In order to show that $(M_r^{\mu_L}, \mathcal{F}_r^{\mu_L}, \omega_r^{\mu_L})$ is equivalent to $(M^L, \mathcal{F}^L, \omega^L)$, at least for r small enough, we need some preliminary results.

Lemma 5. *Let (P, Ω) be a symplectic manifold, H a hypersurface, and Y a symplectic vector field transversal to H and $f \in C^\infty(M, \mathbb{R}^+)$. The hypersurfaces $H_t := \varphi_t^{fY}(H)$ are transversal to Y and hence inherit the structure $(H_t, \mathcal{F}_t, \omega_t)$ of a 2-calibrated foliation.*

The diffeomorphisms φ_t^{fY} preserve the underlying Poisson structures.

Proof. For all t the symplectic leaves of $(H_t, \mathcal{F}_t, \omega_t)$ are the intersection of H_t with the leaves of $\mathcal{D}$, the distribution integrating $\mathrm{Ann}(Y)^\Omega$.

The flow lines of fY are contained in $\mathcal{D}$, therefore $\varphi_t^{fY}: H \to H_t$ sends $\mathcal{F}$ to $\mathcal{F}_t$.

Let $\omega_{\mathcal{D}}$ be the restriction of ω to the leaves of $\mathcal{D}$. From $d\omega = 0, \mathcal{L}_Y \omega = 0$ and $i_Y \omega_{\mathcal{D}} = 0$ we deduce

$$\mathcal{L}_{fY} \omega_{\mathcal{D}} = 0$$

□

Let $(M, \mathcal{F}, \omega)$ be a 2-calibrated foliation (M possibly non-compact) with a riemannian metric g. Let $L \subset M$ be a compact submanifold with $H^1(L; \mathbb{R}) = 0$. Denote by $\mathcal{F}_L$ the leaf containing L. For any $\lambda > 0$ let $T_L(\lambda)$ be the tubular neighborhood of radius λ in $\mathcal{F}_L$ for the leaf metric. We say that λ is admissible if $T_L(\lambda)$ is diffeomorphic to the normal bundle of L in $\mathcal{F}_L$. Fix R any no-where vanishing vector field in $\mathrm{Ker}\omega$. Since $H^1(L; \mathbb{R}) = 0$, by Thurston's stability theorem the foliation in a neighborhood of L is trivial. Therefore, it makes sense to ask R to preserve $\mathcal{F}$.

As in equation 14 and for $\epsilon > 0$ small enough, let

$$T_L(\lambda, \epsilon) = \varphi^R([-\epsilon, \epsilon] \times T_L(\lambda))$$

We say that $\lambda, \epsilon > 0$ are admissible if λ is admissible and $\varphi^R{}_{|[-\epsilon,\epsilon] \times T_L(\lambda)}$ is a diffeomorphism. The neighborhood $T_L(\lambda, \epsilon)$, for $\lambda, \epsilon > 0$ admissible, is endowed with a metric g_R which is the result of restricting g to $T_L(\lambda)$, declaring R to be orthogonal to $T_L(\lambda)$, and then pushing it by φ_t^R

We also have the annular regions

$$\begin{aligned} A_L(\lambda, \lambda') &:= T_L(\lambda) \backslash T_L(\lambda')\ , \lambda > \lambda' \geq 0,\ \lambda \text{ admissible} \\ A_L(\lambda, \lambda', \epsilon, \epsilon') &:= T_L(\lambda, \epsilon) \backslash T_L(\lambda', \epsilon')\ , \lambda > \lambda' \geq 0,\ \epsilon > \epsilon' \geq 0,\ \lambda, \epsilon \text{ admissible} \end{aligned}$$

There is a notion of exactness for Poisson structures and for Poisson morphism between exact Poisson manifolds. For regular Poisson structures $(M, \mathcal{F}, \omega_{\mathcal{F}})$ with codimension one leaves and product foliation -as it will be the case for our local applications- it is equivalent to asking for the existence of a leafwise 1-form $\alpha_{\mathcal{F}}$ such that $d_{\mathcal{F}} \alpha_{\mathcal{F}} = \omega_{\mathcal{F}}$. For two such Poisson manifolds a Poisson morphism $\phi\colon (M, \mathcal{F}, d_{\mathcal{F}} \alpha_{\mathcal{F}}) \to (M', \mathcal{F}', d_{\mathcal{F}'} \bar{\alpha}_{\mathcal{F}'})$ is exact if for each leaf of $\mathcal{F}$ the cohomology class $[\alpha_{\mathcal{F}} - \phi^* \bar{\alpha}_{\mathcal{F}'}]$ is vanishing.

Lemma 6. *Let $(M, \mathcal{F}, \omega)$ and L as above, with $\omega_{\mathcal{F}} = d_{\mathcal{F}} \alpha_{\mathcal{F}}$ exact. Let $\lambda', \epsilon' > 0$ admissible. Suppose $\phi_s\colon T_L(\lambda', \epsilon') \to M$, $s \in [0, 1]$, is a smooth isotopy with $\phi_0 = \mathrm{Id}$ and ϕ_s an exact Poisson map that fixes each leaf setwise. Fix any λ, ϵ admissible with $\lambda > \lambda'$ and $\epsilon > \epsilon'$. Then there exists a constant $\delta(\lambda, \lambda')$ such that if $|\phi_s|_{C^0(T_L(\lambda, \epsilon), g_R)} \leq \delta(\lambda, \lambda')$, there exists an isotopy*

$$\psi_s\colon M \to M$$

such that:

(1) $\psi_0 = \mathrm{Id}$ *and* ψ_s *is a Poisson morphism fixing each leaf setwise.*
(2) *The restriction of* ψ_s *to* $T_L(\lambda', \epsilon')$ *equals* ϕ_s.
(3) *The support of* ψ_s *is contained in the interior of* $T_L(\lambda, \epsilon)$.

Proof. This is a standard result for symplectic manifolds, i.e. for $(\mathcal{F}_L, \omega_{\mathcal{F}})$ and $\epsilon' = \epsilon = 0$, whose proof we sketch.

The first step is extending ϕ_s to an isotopy $\tilde{\phi}_s$ with support in $T_L(\lambda)$ and $\tilde{\phi}_0 = \mathrm{Id}$; this is possible if $\phi_s(T_L(\lambda'))$ is contained in the interior of $T_L(\lambda)$, which is granted by an apprpriate upper bound on the C^0-norm.

From now on we restrict our attention to $T_L(\lambda)$. Let $\omega_s := \tilde{\phi}_s^*\omega$, $\alpha_s := \tilde{\phi}_s^*\alpha$ and $\varsigma_s := d/ds\omega_s$. By hypothesis $\omega = d\alpha$ and $[\alpha_s - \alpha] = 0$ in $H^1(T_L(\lambda'))$. Therefore there exists a smooth family $f_s \in C^\infty(T(\lambda'))$ with $\alpha_s - \alpha = df_s$ in $T_L(\lambda')$.

Extend f_s to a smooth family $F_s \in C^\infty(T_L(\lambda))$ with support in the interior of $T_L(\lambda)$. Define

$$\beta_s = \frac{d}{ds}(\alpha_s - dF_s) \in \Omega^1(T_L(\lambda))$$

By construction β_s is supported in the interior of $A_L(\lambda, \lambda')$ and $d\beta_s = \varsigma_s$. We apply Moser's trick and consider $\xi_s \colon T_L(\lambda) \to T_L(\lambda)$ to be the isotopy associated to the vector fields of Z_s defined by the equation

$$i_{Z_s}\omega_s = -\beta_s$$

Then $\psi_s := \xi_s \circ \tilde{\phi}_s$ is the desired solution.

In the case of a 2-calibrated manifold we proceed similarly. Let t be the parameter of the the interval $[-\epsilon, \epsilon]$. First fix an extension $\tilde{\phi}_s$ with support in the interior of $T_L(\lambda, \epsilon)$. Its existence is granted by a bound on the C^0-norm associated to g_R; it is measured leafwise, and therefore the bound does not depend on ϵ at all.

Consider the leafwise 2-forms $\omega_{\mathcal{F},s} := \tilde{\phi}_s^*\omega_{\mathcal{F}}$ (resp. 1-forms $\alpha_{\mathcal{F},s} := \tilde{\phi}_s^*\alpha_{\mathcal{F}}$). Rewrite them as the 2-parameter family $\omega_{s,t} \in \Omega^2(T_L(\lambda))$ (resp. $\alpha_{s,t} \in \Omega^1(T_L(\lambda))$. We apply the previous construction with the extra parameter t and the result follows; we only need to extend the functions $f_{t,s} \in C^\infty(T_L(\lambda'))$, $t \in [-\epsilon', \epsilon']$, $f_{t,s} = f_s \in C^\infty(T_L(\lambda', \epsilon'))$, to functions $F_s \in C^\infty(T_L(\lambda, \epsilon))$ vanishing near the boundary. $\square$

Lemma 7. *If* $\phi_s \colon A_L(\lambda', \lambda'', \epsilon', \epsilon'') \to M$ *is as in lemma 6, has small enough* C^0*-norm and* $n > 1$*, we can extend it to an isotopy by Poisson maps* $\psi_s \colon A_L(\lambda, \lambda'', \epsilon, \epsilon'') \to M$ *with support in the interior of* $T_L(\lambda, \epsilon)$.

Proof. The only delicate point is constructing a smooth family $f_s \in C^\infty(A_L(\lambda', \lambda'', \epsilon', \epsilon''))$ such that $\phi_s^*\alpha_{\mathcal{F}} - \alpha_{\mathcal{F}} = d_{\mathcal{F}} f_s$ in $A_L(\lambda', \lambda'', \epsilon', \epsilon'')$, because when we consider the parameter $t \in [-\epsilon', \epsilon']$, the domain is $T_L(\lambda')$ if $t \in [-\epsilon', -\epsilon''] \cup [\epsilon'', \epsilon']$, and $A_L(\lambda', \lambda'')$ if $t \in [-\epsilon'', \epsilon'']$. This construction is possible when $n > 1$ because $A_L(\lambda', \lambda'')$ is connected. Then we can select a (finite) good cover $\mathcal{U}$ of $A_L(\lambda', \lambda'')$ and a (finite) good cover $\mathcal{V}$ of $T_L(\lambda'')$ (actually of a small enough tubular neighborhood of $T_L(\lambda'')$), such that the union $\{\mathcal{U}, \mathcal{V}\}$ is a good cover of $T_L(\lambda')$. Hence we can order the elements of $\mathcal{U}$ so that a subset U_i has non-empty connected intersection with the union of the precedent ones; this ordering is extended to subsets of $\mathcal{V}$ with the same property, and we get an ordering $\prec$ of $\{\mathcal{U}, \mathcal{V}\}$ such that $U \prec V$ for every $U \in \mathcal{U}$, $V \in \mathcal{V}$. The subsets $U_i \times [-\epsilon', \epsilon'], V_j \times [-\epsilon', -\epsilon''], V_j \times [\epsilon'', \epsilon']$, for every $U_i \in \mathcal{U}, V_j \in \mathcal{V}$, are a good cover $\{\mathcal{U}, \mathcal{V}^-, \mathcal{V}^+\}$ of $A_L(\lambda', \lambda'', \epsilon', \epsilon'')$. It carries an induced ordering as the result of identifying $\{\mathcal{U}, \mathcal{V}\}$ with $\{\mathcal{U}, \mathcal{V}^-\}$ and pushing the ordering $\prec$ to the latter, then identifying again $\mathcal{V}$ with $\mathcal{V}^+$ and pushing the ordering of $\mathcal{V}$, and finally declaring any element of $\{\mathcal{U}, \mathcal{V}^-\}$ to precede any element of $\mathcal{V}^+$.

By construction, any subset has non-empty connected intersection with the union of the precedent ones.

The functions f_s we look for are constructed using the (leafwise) Poincarè lemma with an initial choice for the first subset of the ordered covering of $A_L(\lambda', \lambda'', \epsilon', \epsilon'')$. $\square$

Remark 7. *Suppose that we have $L_r \subset (M_r, \mathcal{F}_r, d\alpha_{\mathcal{F},r})$, $T_{L_r}(\lambda)$, R_r, $r \in (0, r_0]$, smooth families of embedded of submanifolds, tubular neighborhoods of them and tranverse vector fields as above, with $\alpha > 0$ admissible for all r. Let $\epsilon > 0$ be also admissible for all r. With the aforementioned choices all submanifolds $T_{L_r}(\alpha, \epsilon)$ are canonically isomorphic. Construct the metric g_R in $T_{L_{r_0}}(\alpha, \epsilon)$ as shown at the beginning of this subsection, as use the isomorphism to transfer it to any $T_{L_r}(\alpha, \epsilon)$.*

Let

$$\phi_{r,s} \colon (A_{L_r}(\lambda', \lambda'', \epsilon'_r, \epsilon''_r), d\alpha_{\mathcal{F},r}) \to (T_{L_r}(\lambda, \epsilon), \alpha_{\mathcal{F},r}),$$

be a smooth family of isotopies by exact Poisson morphisms, where the radius $\epsilon'_r, \epsilon''_r$ also vary smoothly (for example they can be an strictly monotone sequence converging to zero, as it will be the case for our application). Assume that for any $\delta > 0$ there exist $r(\delta)$ such that

$$|\phi_{r,s}|_{C^0(T_{L_r}(\lambda,\epsilon),g_R)} \leq \delta$$

for all $r \leq r(\delta)$. Then there exists r' such that for any $r \leq r'$ we have

$$\psi_{r,s} \colon A_{L_r}(\lambda, \lambda'', \epsilon, \epsilon''_r) \to M$$

an smooth extension by Poisson morphism of $\phi_{r,s}$, with support in the interior of $A_{L_r}(\lambda, \lambda'', \epsilon, \epsilon''_r)$ and with $\psi_{r,0} = \mathrm{Id}$ (here the smoothness is on s for each fixed r; we make no statement about the dependence on r of the extensions).

This follows automatically from lemma 7. Notice that after we use the natural identifications, the dependence on r only appears in ϵ'_r and ϵ''_r, but this does not affect at all for the constant $\delta(\lambda, \lambda')$ does not depend on them.

3.4. **Lagragian surgery equals generalized Dehn surgery.** We start by proving two preliminary lemmas that will allow us to apply the remark of the previous subsection.

Let us go back to the symplectic (n+1)-handle. For each $r \in [-r_0, 0) \cup (0, r_0]$ and for $\lambda > 0$ admissible we have defined neighborhoods $T_r(\lambda)$. Using the flow of R'_r we get for $\lambda, \epsilon > 0$ admissible the subsets $T_r(\lambda, \epsilon)$ and the annular regions $A_r(\lambda, \lambda', \epsilon, \epsilon')$, for $\lambda > \lambda' \geq 0$, $\epsilon > \epsilon' \geq 0$ (the annular regions also defined for $r = 0$).

According to the previous subsection the neighborhoods $T_r(\lambda, \epsilon) \backslash \Sigma_r$ carry a metric g_R, coming from the canonical metric of $(T(\lambda), d\alpha_{\mathrm{can}})$ (defined by the round metric in S^n). This metric *does not depend on* r. The restriction of the Euclidean metric defines another metric g_0 (changing with r).

Lemma 8.

1. *For any λ, epsilon admissible and any $\lambda' > 0$, $\lambda > \lambda'$, the restriction of g_R and g_0 to $A_r(\lambda, \lambda', \epsilon, 0)$ are comparable metrics, and the comparison constants do not depend on r.*
2. *Let $f \in C^\infty(\mathbb{C}^{n+1} \backslash \{0\})$, $f > 0$, such that $h_* f Y_1 = -\partial/\partial x$ and define for $r \in [0, r_0]$*

$$F_r(\lambda, \lambda', \epsilon, 0) := \bigcup_{t \in [0, 2r]} \varphi_t^{fY_1} A_r(\lambda, \lambda', \epsilon, 0)$$

Then there exist $r_1 > 0$ such that for some $C' > 0$

$$||y|| \geq C',\ \forall y \text{ in the closure of } \bigcup_{r\in[0,r_1]} F_r(\lambda,\lambda',\epsilon,0) \tag{26}$$

Proof. Let $f \in C^\infty(\mathbb{C}^{n+1}\backslash\{0\})$, $f > 0$, such that $h_* f Y_1 = -\partial/\partial x$. Observe that for some $C > 0$,

$$||y|| \geq C,\ \forall y \text{ in the closure of } \bigcup_{r\in[-r_0,r_0]} A_r(\lambda,\lambda') \tag{27}$$

Therefore the distance of the closure of $\bigcup_{r\in[-r_0,r_0]} A_r(\lambda,\lambda',\epsilon,0)$ to the origin is strictly positive, and point 1 of the lemma follows.

Point 2 is a consequence of equation 27. □

For any $z \in \bar{D}(r_0)$, let α_z denote the restriction to the 1-form $\alpha_{\mathbb{C}^{n+1}} + \zeta$ to the symplectic fiber $h^{-1}(z)$. We recall the following standard result:

Lemma 9. *Let $\gamma \subset \bar{D}(r_0)$ be any immersed curve avoiding the origin and let ρ_γ denote the parallel transport w.r.t $\Omega_{\mathbb{C}^{n+1}} + d\zeta$ along γ. Then if $n > 1$*

$$\rho_\gamma\colon (h^{-1}(\gamma(0)), d\alpha_{\gamma(0)}) \to (h^{-1}(\gamma(1)), d\alpha_{\gamma(1)})$$

is an exact symplectomorphism.

If γ crosses through the origin (only once, say) then away from lagrangian spheres

$$\rho_\gamma\colon (h^{-1}(\gamma(0))\backslash\Sigma_{\gamma(0)}, d\alpha_{\gamma(0)}) \to (h^{-1}(\gamma(1))\backslash\Sigma_{\gamma(1)}, d\alpha_{\gamma(1)})$$

is an exact symplectomorphism.

Let $L \subset (M,\mathcal{F},\omega)$ be a parametrized lagrangian sphere. Extend it to a symplectomorphism

$$\varphi\colon (U,\omega_{\mathcal{F}}) \to (T(\lambda), d\alpha_{\text{can}}) \tag{28}$$

and perform the lagrangian surgery to obtain the 2-calibrated foliations $(M_r^{\mu_L},\mathcal{F}_r^{\mu_L},\omega_r^{\mu_L})$, for $r \in (0,r_0]$ and for some $\lambda > 0$.

Let $(M^L,\mathcal{F}^L,\omega^L)$ be the result of performing generalized Dehn surgery with the fixed identification φ of equation 28, and using a model Dehn twist supported in the interior of $T(\lambda/7)$.

Theorem 3. *Under the assumption $n > 1$, there exists $r' > 0$ such that for all $r \in (0,r']$ we have equivalences of 2-calibrated foliations*

$$\phi_r\colon (M^L,\mathcal{F}^L,\omega^L) \to (M_r^{\mu_L},\mathcal{F}_r^{\mu_L},\omega_r^{\mu_L}) \tag{29}$$

Proof. For all $r \in (0,r_0]$ small enough we will construct an exhaustion $W_r^0 \subset W_r^1 \subset W_r^2 \subset W_r^3 = M^L$ by open sets, and define ϕ_r in four stages by extending it from one subset of the exhaustion to the following one.

From now on we assume that we work in the symplectic (n+1)-handle and use the notation of proposition 4.

Stage 1. Let $T_r(\lambda,\epsilon) \subset (M,\mathcal{F},\omega)$. Its complement $(M,\mathcal{F},\omega)\backslash T_r(\lambda,\epsilon)$ can be seen as a subset of both $(M,\mathcal{F},\omega)$ and $(M_r^{\mu_L},\mathcal{F}_r^{\mu_L},\omega_r^{\mu_L})$.

A subset of $M_r^{\mu_L}$ is the image of $A_r(\lambda,5\lambda/6,\epsilon,5\epsilon/6)$ by time one flow of the vector field $f_r Y_1$, where f_r is the function of equation 24.

Our first subset is $W_r^0 := M\backslash T_r(5\lambda/6,5\epsilon/6)$, where we define

$$\phi_r = \begin{cases} \mathrm{Id} & \text{in } M\backslash T_r(\lambda,\epsilon) \\ \varphi_1^{f_r Y_1} & \text{in } A_r(\lambda,5\lambda/6,\epsilon,5\epsilon/6) \end{cases}$$

By lemma 5 this diffeomorphism preserves the Poisson structures.

Stage 2. For any $r \in [0,r_0]$ fix a function $\kappa_r\colon [-\epsilon,\epsilon] \to [-\epsilon,\epsilon]$ such that

- κ_r is smooth in both variables (r,t).
- $\kappa_r(-t) = -\kappa_r(t)$ and κ_r is monotone increasing.
- $\kappa_{r|[-\epsilon_r,\epsilon_r]} = 0$, with ϵ_r smooth and converging to zero when $r \searrow 0$ (so $\kappa_0 = 0$).

Let $h_t(c,d)$ denote the horizontal segment joining the points (c,t) with (d,t). Consider the function

$$\begin{aligned} \tilde{\phi}_r \colon (A_r(\lambda,0,\epsilon,\epsilon_r), d\alpha_{\mathcal{F},r}) &\longrightarrow (H_r^{\mu_L,0}, d\alpha_{\mathcal{F},-r}) \\ y &\longmapsto \rho_{v_{-r}(\kappa_{-r}(t),t)} \circ \rho_{h_t(-r,r)} \circ \rho_{v_r(t,\kappa_r(t))}(y) \end{aligned} \tag{30}$$

where y belongs to the t-leaf of $A_r(\lambda,0,\epsilon,\epsilon_r)$, $t \in [-\epsilon,\epsilon]$. By lemma 9 it is an exact Poisson morphism.

Let $W_r^1 := W_r^0 \cup A_r(5\lambda/6, \lambda/2, 5\epsilon_r/6, 5\epsilon_r/6)$. Our aim is to find another Poisson morphism in W_r^1 extending ϕ_r and the restriction of $\tilde{\phi}_r$ to $A_r(2\lambda/3, \lambda/2, 2\epsilon_r, 3\epsilon_r/2)$, and we will do it by applying remark 7.

To connect ϕ_r and $\tilde{\phi}_r$ by an isotopy of exact Poisson morphism, just notice that ϕ_r is defined as in equation 30 but using $\kappa_r = \mathrm{Id}$.

Therefore if we interpolate smoothly between both functions by monotone increasing functions $\kappa_{r,s}$, we obtain

$$\phi_{r,s} \colon A_r(\lambda,0,\epsilon,\epsilon_r) \to H_r^{\mu_L,0}$$

with $\phi_{r,0} = \phi_r$, $\phi_{r,1} = \tilde{\phi}_r$.

Define the exact Poisson morphisms

$$\tilde{\sigma}_{r,s} := \phi_r^{-1} \circ \phi_{r,s} \colon (A_r(2\lambda/3, \lambda/2, 2\epsilon_r, 3\epsilon_r/2), d\alpha_{\mathcal{F},r}) \to (T_r(\lambda,\epsilon), d\alpha_{\mathcal{F},r})$$

Notice that equation 26 implies that the parallel transport involved in the definition of $\phi_{r,s}$ occurs away from a neighborhood of the critical point. Therefore, by shrink the length of the curves in $\mathbb{C}$ over which we translate, i.e. by making r (and hence ϵ_r) small enough, we can make $||y - \phi_{r,t}(y)||$ arbitrarily small. Thus for any $\delta > 0$ a constant $r'(\delta) > 0$ exists such that for every $r \leq r'$ we have

$$||\tilde{\sigma}_{r,s}(y) - y|| \leq \delta, \; y \in A_r(2\lambda/3, \lambda/2, 2\epsilon_r, 3\epsilon_r/2)$$

Notice that $||\tilde{\sigma}_{r,s}(y) - y|| = |\tilde{\sigma}_{r,s}(y)|_{C^0,g_0}$. By point 1 in lemma 8 we have

$$|\tilde{\sigma}_{r,s}|_{C^0,g_{\mathrm{R}}} \leq C''\delta,$$

for some C'' independent of r, and therefore we are in the hypothesis of remark 7. Let

$$\tilde{\psi}_{r,s} \colon A_r(5\lambda/6, \lambda/2, 5\epsilon/6, 3\epsilon_r/2) \to A_r(5\lambda/6, \lambda/2, 5\epsilon/6, \epsilon_r)$$

be the isotopy furnished by the aforementioned remark. Then

$$\tilde{\phi}_{r,s} := \phi_r \circ \tilde{\psi}_{r,s} \colon (A_r(5\lambda/6, \lambda/2, 5\epsilon/6, 3\epsilon_r/2), d\alpha_{\mathcal{F},r}) \to (M_r^{\mu_L,0}, d\alpha_{\mathcal{F},-r})$$

is an isotopy of Poisson morphism and $\tilde{\phi}_{r,1}$ interpolates between ϕ_r and $\tilde{\phi}_r$. The map

$$\phi_r = \begin{cases} \phi_r & \text{in } \; W_r^0 \\ \tilde{\phi}_{r,1} & \text{in } \; A_r(5\lambda/6, \lambda/2, 5\epsilon/6, 3\epsilon_r/2) \end{cases} \tag{31}$$

is a well defined Poisson morphism.

Stage 3. Extend to $W_r^2 := W_r^1 \cup A_r(\lambda/2, \lambda/6, 3\epsilon_r/2, \epsilon_r)$.

Consider the arcs

$$\begin{aligned} \partial\bar{D}^{+}(t,r) &:= \{(0,t)+re^{i\theta}\,|\,0\leq\theta\leq\pi\} \\ \partial\bar{D}^{-}(t,r) &:= \{(0,t)+re^{i\theta}\,|\,0\geq\theta\geq-\pi\} \end{aligned}$$

Define a new exact Poisson morphism

$$\begin{aligned} \hat{\phi}_r^{+}\colon (T_r^{+}(\lambda/2,2\epsilon_r),d\alpha_{\mathcal{F},r}) &\longrightarrow (H_r^{\mu_L,0},d\alpha_{\mathcal{F},-r}) \\ y &\longmapsto \rho_{v_{-r}(\kappa_r(t),t)}\circ\rho_{\partial\bar{D}^{+}(\kappa_r(t),r)}\circ\rho_{v_r(t,\kappa_r(t))}(y) \end{aligned} \tag{32}$$

where y belongs to the t-leaf $\varphi_t^{R'_r}(T_r(\lambda/2))$ and $t\geq 0$. For t-leafs with $t\leq 0$ we define $\hat{\phi}_r^{-}$ using $\rho_{\partial\bar{D}^{-}(\kappa_r(t),r)}$ instead of $\rho_{\partial\bar{D}^{+}(\kappa_r(t),r)}$.

Let

$$\hat{\phi}_r := \begin{cases} \hat{\phi}_r^{+} & \text{in } T_r^{+}(\lambda/2,2\epsilon_r) \\ \hat{\phi}_r^{-} & \text{in } T_r^{-}(\lambda/2,2\epsilon_r) \end{cases}$$

We claim that its restriction to $A_r(\lambda/2,\lambda/6,2\epsilon_r,\epsilon_r)$ is well defined and smooth. For $t\in[-\epsilon_r,\epsilon_r]$ we use the rescaled local Reeb vector fields R'_r to see both $\hat{\phi}_r^{+}$ and $\hat{\phi}_r^{-}$ as a family of maps $\hat{\phi}_{r,t}^{+},\hat{\phi}_{r,t}^{-}\colon A_r(\lambda/2,\lambda/6)\to T_{-r}(\lambda)$.

Since the functions κ_r shrink the intervals $[-\epsilon_r,0]$ and $[0,\epsilon_r]$ to the origin, the two aforementioned families of maps do not depend on t. Therefore $\hat{\phi}_r$ is well defined and smooth if and only if

$$\hat{\phi}_{r,0}^{+}=\hat{\phi}_{r,0}^{-} \text{ in } A_r(\lambda/2,\lambda/6) \tag{33}$$

By construction $(\hat{\phi}_{r,0}^{-})^{-1}\circ\hat{\phi}_{r,0}^{+}=\rho_{\partial\bar{D}(r)}$, and the generalized Dehn twist we chose $\tau_r=\varphi_r\circ\rho_{\partial\bar{D}(r)}\circ\varphi_r^{-1}\colon T(\lambda)\to T(\lambda)$ has support in $T(\lambda/7)$, and hence equation 33 holds.

We need to connect $\hat{\phi}_r$ and ϕ_r in equation 41 in the annular region $A_r(\lambda/2,\lambda/6,2\epsilon_r,\epsilon_r)$, and this is done using the families

$$\phi_{r,s}^{+}\colon (A_r(\lambda/2,\lambda/6,2\epsilon_r,\epsilon_r),d\alpha_{\mathcal{F},r})\longrightarrow (H_r^{\mu_L,0},d\alpha_{\mathcal{F},-r}) \tag{34}$$

defined by the rule

$$y\longmapsto \rho_{v_{-r}(\kappa_r(t),t)}\circ\rho_{h_{\kappa_r(t)}(-sr,-r)}\circ\rho_{\partial\bar{D}^{+}(\kappa_r(t),sr)}\circ\rho_{h_{\kappa_r(t)}(r,sr)}\circ\rho_{v_r(t,\kappa_r(t))}(y)$$

for y in the t-leaf, $t\geq 0$, and $\phi_{r,s}^{-}$ (using $\rho_{\partial\bar{D}^{-}(\kappa_r(t),sr)}$) for negative values of t.

We claim that

$$\phi_{r,s} := \begin{cases} \phi_{r,s}^{+} & \text{in } A_r(\lambda/2,\lambda/6,2\epsilon_r,\epsilon_r) \\ \phi_{r,s}^{-} & \text{in } A_r(\lambda/2,\lambda/6,2\epsilon_r,\epsilon_r) \end{cases}$$

is a well defined smooth isotopy by exact Poisson morphisms. We start by checking that $\phi_{r,s}$ is well defined (and smooth) for each $s\in[0,1]$. As before, we use the local vector fields R'_r to see each Poisson morphism as a family $\phi_{r,s,t}^{+}$ and $\phi_{r,s,t}^{-}$, which for the same reasons do not depend on t.

The equality

$$\phi_{r,s,0}^{+}=\phi_{r,s,0}^{-} \text{ in } A_r(\lambda/2,\lambda/6)$$

is equivalent to

$$\rho_{h_0(-r,-sr)}\circ\phi_{r,s,0}^{+}\circ\rho_{h_0(sr,r)}=\rho_{h_0(-r,-sr)}\circ\phi_{r,s,0}^{-}\circ\rho_{h_0(sr,r)} \text{ in } A_{sr}(\lambda/2,\lambda/6) \tag{35}$$

If we compose with the the inverse of the r.h.s, then 35 transforms into

$$\mathrm{Id}=\rho_{\partial\bar{D}(sr)} \text{ in } A_{sr}(\lambda/2,\lambda/6),$$

which holds because again the support of τ_{sr} is contained in $T(\lambda/7)$.

Lemma 9 implies that each $\phi_{r,s}$ is an exact Poisson morphism.

The smoothness of $\phi_{r,s}$ is not straightforward when $s=0$, but it holds because of the smoothness of the family $\partial\bar{D}^+(sr)$ in s. We start by observing that since for small values of t the families $\phi^+_{r,s,t}$ and $\phi^-_{r,s,t}$ are constant in t and equal, the smoothness follows from the smoothness of both $\phi^+_{r,s,t}$ and $\phi^-_{r,s,t}$.

Let $X := \partial/\partial x$, $Y := \partial/\partial y$ and $\Theta_t =: xdy-(y-t)dx$, $t\in\mathbb{R}$, vector fields on $\mathbb{R}^2$.

Let $\tilde{X},\tilde{Y},\tilde{\Theta}_t \in \mathfrak{X}(\mathbb{C}^{n+1}\backslash\{0\})$ be their horizontal lifts w.r.t. $\Omega_{\mathbb{C}^{n+1}}+d\zeta$. The flows $\varphi_l^{\tilde{X}}$, $\varphi_l^{\tilde{Y}}$, $\varphi_l^{\tilde{\Theta}_t}$ are smooth (in t,l). One checks

$$\phi^+_{r,s}(y) = \varphi^{\tilde{Y}}_{t-\kappa_r(t)}\circ\varphi^{\tilde{X}}_{(1-s)r}\circ\varphi^{\tilde{\Theta}_{\kappa_r(t)}}_{\pi}\circ\varphi^{-\tilde{X}}_{(1-s)r}\circ\varphi^{-\tilde{Y}}_{t-\kappa_r(t)}(y), \tag{36}$$

and this proves the claim.

Thus for small enough values of r we get $\hat{\phi}_{r,1}\colon A_r(\lambda/2,\lambda/6,3\epsilon_r/2,\epsilon_r)\to H_r^{\mu_L,0}$ matching $\hat{\phi}_r$ in $A_r(\lambda_2,\lambda/6,2\epsilon_r,\epsilon_r)$, and such that

$$\phi_r =: \begin{cases}\phi_r & \text{in } W_r^1\\ \hat{\phi}_{r,1} & \text{in } A_r(\lambda/2,\lambda/6,3\epsilon_r/2,\epsilon_r)\end{cases} \tag{37}$$

is a Poisson morphism.

Stage 4: Cut open and extend to $(M^L,\mathcal{F}^L,\omega^L)$.

Let $(M^{\text{open}}_{T_r(\lambda/6)},\mathcal{F}^{\text{open}},\omega^{\text{open}})$ the result of cutting M open along $T_r(\lambda/6)$. We have $M^{\text{open}}_{T_r(\lambda/6)} = W_r^2\cup T_r^+(\lambda/6,\epsilon_r)\cup T_r^-(\lambda/6,\epsilon_r)$

Consider the map

$$\phi_r^{\text{open}} = \begin{cases}\phi_r & \text{in } W_r^2\\ \hat{\phi}_r^+ & \text{in } T_r^+(\lambda/6,\epsilon_r)\\ \hat{\phi}_r^- & \text{in } T_r^-(\lambda/6,\epsilon_r)\end{cases}$$

Equation 37 implies that ϕ_r^{open} is a well defined Poisson Morphism. Recall that if we use the local Reeb vector fields R'_r, which have *negative co-orientation*, the maps $\hat{\phi}^+_{r,t}$ and $\hat{\phi}^-_{r,t}$ are independent of t, and $\left(\hat{\phi}^-_{r,0}\right)^{-1}\circ\hat{\phi}^+_{r,0}=\rho_{\partial\bar{D}(r)}$.

By construction

$$(M^{\text{open}}_{T_r(\lambda/6)},\mathcal{F}^{\text{open}},\omega^{\text{open}})/T_r^+(\lambda/6)\sim\rho_{\partial\bar{D}(r)}(T_r^-(\lambda/6))\cong(M^L,\mathcal{F}^L,\omega^L),$$

where the *Poisson equivalence* uses the fact that $(M^L,\mathcal{F}^L,\omega^L)$ does not depend either on the generalized Dehn twist τ_r, $r\in(0,r_0]$, or in the parametrization φ_r of equation 28.

Therefore $\phi_r\colon M^{\text{open}}_{T_r(\lambda/6)}\to M_r^{\mu_L}$ descends to a Poisson equivalence

$$\phi_r\colon(M_r^{\mu_L},\mathcal{F}_r^{\mu_L},\omega_r^{\mu_L})\to(M^L,\mathcal{F}^L,\omega^L)$$

and the theorem is proven.

By the general position arguments used in proposition 2, since $n>1$ this Poisson equivalence is an equivalence of 2-calibrated foliations. □

Remark 8. *Similarly, for $n>1$ and every $r>0$ small enough one constructs equivalences*

$$\phi_r\colon(M_r^{-\mu_L},\mathcal{F}_r^{-\mu_L},\omega_r^{-\mu_L})\to(-M^{L^-},\mathcal{F}^{L^-},\omega^{L^-})$$

Let Ω be a symplectic form defined in a small (contractible) neighborhood of the origin, such that $\Omega(0)$ is of type (1,1) and positive (Kahler at the origin). Then Ω is *compatible with the function h*, meaning that the fibers of h are symplectic w.r.t. Ω (perhaps in a smaller neighborhood of the origin). Moreover, if Ω,Ω' are two such symplectic forms, then its convex combination defines a family Ω_t, $t\in[0,1]$, of symplectic forms compatible with h -because all are Kahler at the

origin- connecting the given ones. Whenever we have such a family, being the neighborhood W contractible, we will fix $\alpha_t \in \Omega^1(W)$ a smooth family such that $d\alpha_t = \Omega_t$.

We claim that lagrangian surgery can be defined using any symplectic form Ω Kahler at the origin.

Lemma 10. *Let Ω_t, $t \in [0,1]$ be a family of (perhaps local) symplectic forms Kahler at the origin. Let $Y_{\Omega_t} \in \mathfrak{X}(W\backslash\{0\})$ be the hamiltonian w.r.t. Ω_t of $-\mathrm{Im}h$.*

1. *Y_{Ω_t} is a section of $\mathrm{Ann}(Y)^{\Omega_t}$.*
2. *$h_*Y_{\Omega_t}$ is an strictly negative multiple of $\partial/\partial x$.*
3. *Y_{Ω_t} has a non-degenerate singularity at the origin with n+1 positive eigenvalues and n+1 negative eigenvalues.*
4. *For each $z \in \mathbb{C}\backslash\{0\}$ we have lagrangian spheres $\Sigma_{\Omega_t,z} \subset h^{-1}(z)$ characterized as the set of points contracting into the critical point by the parallel transport over the radial segments; the spheres come with a parametrization up to isotopy and the action of O(n+1) (they are "framed") that can be chosen smoothly on t. More generally, for each z and γ an embedded curve joining z and the origin, the points over z sent to the origin by parallel transport over γ are lagrangian spheres which depend smoothly on both t and γ.*

Proof. This a generalization of lemma 1.13 in [37] for local symplectic forms which are not Kahler but at the origin.

Points 1,2,3 are a straightforward calculation.

Point 3 implies that Y_{Ω_t} is a smooth family of hyperbolic vector fields. Point 2 implies that $h^{-1}(h_0(0,r))$ is an open neighborhood of 0 in the stable manifolds $W^s(Y_{\Omega_t})$. The stable manifold theorem with parameters (see [31], where the proof is seen to depend smoothly on parameters) gives parametrizations

$$\Psi_t^{\mathrm{st}} : B^{n+1}(r) \to W^s(Y_{\Omega_t})$$

of the aforementioned neighborhoods. In particular $\Sigma_{\Omega_t,r}$, $r > 0$, are smooth spheres, and their parametrizations

$$l_t : S^n \to \Sigma_{\Omega_t,r} \tag{38}$$

induced by Ψ_t^{st} are unique up to isotopy and the action of O(n+1) (the latter associated to the choice of an orthonormal basis of the tangent space of $W^s(Y_{\Omega_t})$ at the origin).

That $\Sigma_{\Omega_t,z}$ are lagrangian follow from point 2, exactly as in the proof of lemma 1.13 in [37].

The result for any other point z and the radial segment joining it to the origin, or more generally a curve γ joining it to the origin, follows from the previous ideas applied to the hamiltonian of $-\mathrm{Im}(F \circ h)$, where $F : \mathbb{C} \to \mathbb{C}$ is a diffeomorphism fixing the origin which sends γ to $[0, r_0]$, for some $r_0 > 0$. The smooth dependence on γ follows from choosing diffeomorphisms F_γ with the same smooth dependence. □

For some fixed r_0 small enough we extend

$$l_t : S^n \to \Sigma_{\Omega_t,r_0}$$

to a smooth family of symplectic parametrizations

$$\varphi_{t,r_0} : (U_{r_0}, \Omega_{t|U_{r_0}}) \to (T(\lambda), d\alpha_{\mathrm{can}})$$

We push it over the origin by $\rho_{\Omega_t,h_0(r_0,0)}$ -the parallel transport w.r.t. Ω_t over the horizontal segment $h_0(r_0,0)$- and extend it over $\bar{D}(r_0)$ by using the radial parallel transport again, giving rise to

$$\Phi_{[0,1]}{}^{-1}\colon (\mathbb{C}\times(T\backslash T(0))\times[0,1]\to\mathbb{C}^{n+1}\backslash\bigcup_{z\in\bar{D}(r_0)}\Sigma_{\Omega_t,z} \tag{39}$$

which extends (also smoothly) to T for fibers over $(r,0)$, $r>0$. It restricts for each $t\in[0,1]$ to a diffeomorphism Φ_t, whose restriction to each fiber over $z\in\mathbb{C}$ is a symplectomorphism $\varphi_{t,z}^{-1}$.

We cannot in principle define the inverse of $\Phi_t{}^{-1}$ for all t in a fixed domain, because the subsets $\bigcup_{z\in\bar{D}(r_0)}\Sigma_{\Omega_t,z}$ vary with t. For z in a fixed line through the origin, the corresponding lagrangian spheres together with the origin are the union of the stable and unstable manifold for the hamiltonian of $-\mathrm{Im}(F\circ h)$, where F is the rotation of angle θ. Lemma 10 gives smooth dependence on both t and θ.

Therefore, for any neighborhood W_1 of the origin there exists $r'>0$ such that $\bigcup_{z\in\bar{D}(r')}\Sigma_{\Omega_t,z}\subset W_1$ for all t. Hence, for any fixed $\lambda>0$ and by shrinking W if necessary we get

$$\Phi_{[0,1]}\colon (W\backslash W_1)\times[0,1]\to\bar{D}(r')\times A(\lambda,\lambda/7)$$

such that $\Phi_t=(\Phi_t{}^{-1})^{-1}$.

Another consequence of the smooth dependence on t of the constructions is that lemma 8 holds for all t: for any fixed $\lambda,\lambda'>0$, if λ is small enough it is admissible for all t, and the metrics g_0 and g_R in $A_{t,r}(\lambda,\lambda',\epsilon,0)$ are comparable, the comparison constant being independent of t and $r\in[-r',r']$. It is also true that the closure of the subset $\bigcup_{t\in[0,1]}F_{t,r}(\lambda,\lambda',\epsilon,0)$ -defined as in lemma 8- is at strictly positive distance of the origin.

Now fix Ω any symplectic form Kahler at the origin. The previous construction without parameter gives us for r' small enough a parametrization

$$\Phi\colon \mathbb{C}^{n+1}\backslash\Sigma\to\bar{D}(r')\times T\backslash T(0),$$

where Σ is the union of the lagrangian spheres Σ_z associated to the radial parallel transport. This construction restricts for each $r\in(0,r']$ to a symplectomorphism

$$\varphi_r\colon (h_r^{-1}\cap W,\Omega_{|h_r^{-1}\cap W})\to(T(\lambda),d\alpha_{\mathrm{can}})$$

These are the necessary ingredients to make proposition 4 work but using the symplectic form Ω instead of $\Omega_{\mathbb{C}^{n+1}}+d\zeta$. Therefore, for all $r>0$ small enough we get 2-calibrated foliations $(M^{\mu_L}_{\Omega,r},\mathcal{F}^{\mu_L}_{\Omega,r},\omega^{\mu_L}_{\Omega,r})$.

With these preliminary results we can now prove the main theorem of this subsection.

Theorem 4. *For any symplectic form Ω Kahler at the origin, and under the assumption $n>1$ and $\lambda>0$ small enough, there exists $r''>0$ such that for all $r\in(0,r'']$ we have equivalences of 2-calibrated foliations*

$$\phi_r\colon (M^L,\mathcal{F}^L,\omega^L)\to(M^{\mu_L}_{\Omega,r},\mathcal{F}^{\mu_L}_{\Omega,r},\omega^{\mu_L}_{\Omega,r}) \tag{40}$$

Proof. The proof is a modification of the proof of theorem 3.

We use the symplectic (n+1)-handle associated to Ω and repeat stage 1 word by word.

Let $\Omega_l=l\Omega+(1-l)(\Omega_{\mathbb{C}^{n+1}}+d\zeta)$, $l\in[0,1]$. In stage 2 the Poisson equivalence is extended to

$$\phi_r=\begin{cases}\phi_r & \text{in } W_r^0\\ \tilde{\phi}_r & \text{in } A_r(5\lambda/6,\lambda/2,5\epsilon/6,3\epsilon_r/2)\end{cases} \tag{41}$$

whose definition for y in a t-leaf with $t \in [-\epsilon_r, \epsilon_r]$ is

$$y \mapsto \rho_{\Omega_1, v_{-r}(0,t)} \circ \rho_{\Omega_1, h_0(r,-r)} \circ \rho_{\Omega_1, v_r(t,0)}$$

Consider $\beta_r \colon [-\epsilon_r, \epsilon_r] \to [0,1]$ with $\beta_r(-t) = \beta_r(t)$, $\beta_{r|[0,\epsilon_r/3]} = 0$, $\beta_{r|[2\epsilon_r/3,\epsilon_r]} = 1$. For each y in a t-leaf with $t \in [-\epsilon_r, \epsilon_r]$ define the map

$$y \mapsto \rho_{\Omega_1, v_{-r}(0,t)} \circ \varphi_{1,r}^{-1} \circ \varphi_{\beta_r(t),r} \circ \rho_{\Omega_{\beta_r(t), h_0(-r,r)}} \circ \varphi_{\beta_r(t),r}^{-1} \circ \varphi_{1,r} \circ \rho_{\Omega_1, v_r(t,0)} \qquad (42)$$

The difference is that we make the parallel transport along the horizontal segment w.r.t. a different symplectic form (and to do that we change the parametrization at $h^{-1}(r)$ and $h^{-1}(-r)$).

In order to modify ϕ_r so that in the t-leaves with $t \in [-\epsilon_r, \epsilon_r]$ is the map of equation 42, we observe that ϕ_r admits the same description but using instead of the function β_r, the function which equals 1 everywhere. Therefore, by connecting these two functions with a suitable family $\beta_{r,s}$ we get a family of maps $\phi_{r,s}$.

A consequence of the validity of lemma 8 for all Ω_l uniformly on l, is that the reparametrizations $\varphi_{\beta_{r,s}(t),r}^{-1} \circ \varphi_{1,r}$ and $\varphi_{1,r}^{-1} \circ \varphi_{\beta_{r,s}(t),r}$ have C^0-norm w.r.t. g_R that decreases arbitrarily with r (and the same happens with $\rho_{\Omega_{\beta_{r,s}(t), h_0(-r,r)}}$). Moreover, since $H^1(T_r(\lambda); \mathbb{R}) = H^1(T_{-r}(\lambda); \mathbb{R}) = 0$ they are exact symplectomorphisms and also their restriction to the corresponding annuli (and the maps $\rho_{\Omega_{\beta_{r,s}(t), h_0(-r,r)}}$ are also exact symplectomorphisms because lemma 9 equally holds for arbitrary symplectic forms Kahler at the origin). Thus the maps $\phi_{s,r}$ are exact Poisson morphisms and we can apply remark 7 to obtain the desired Poisson morphism.

We repeat stage 3 in theorem 3 with a similar modification: there we used families of maps which on t-leaves with $t \in [0, \epsilon_r]$ (resp. $[-\epsilon_r, 0]$) defined by

$$y \mapsto \rho_{\Omega_1, v_{-r}(0,t)} \circ \rho_{\Omega_{1, \gamma^+_{r,s,t}}} \circ \rho_{\Omega_1, v_r(t,0)},$$

where $\rho_{\Omega_{1, \gamma^+_{r,s,t}}}$ is parallel translation w.r.t. Ω_1 over a curve $\gamma^+_{r,s,t}$ joining the points $(r,0)$ and $(-r,0)$ (for negative t-leaves we use $\gamma^-_{r,s,t}$).

We use instead

$$y \mapsto \rho_{\Omega_1, v_{-r}(0,t)} \circ \varphi_{1,r}^{-1} \circ \varphi_{\beta_{r,s}(t),r} \circ \rho_{\Omega_{\beta_{r,s}(t), \gamma^+_{r,s,t}}} \circ \varphi_{\beta_{r,s}(t),r}^{-1} \circ \varphi_{1,r} \circ \rho_{\Omega_1, v_r(t,0)}$$

Since parallel transport in the 0-leaf is made w.r.t Ω_0, we conclude as in theorem 3 that our maps are well defined regardless the curve $\gamma^+_{r,s,0}$ or $\gamma^-_{r,s,0}$ chosen in the definition. We can equally apply remark 7 to conclude the existence of the desired perturbation.

Finally step 4 is the same as in theorem 3, and this concludes the proof.

□

Remark 9. *We equally get equivalences*

$$\phi_r \colon (M^{-L}, \mathcal{F}^{-L}, \omega^{-L}) \to (M^{\mu_{L^-}}_{\Omega,r}, \mathcal{F}^{\mu_{L^-}}_{\Omega,r}, \omega^{\mu_{L^-}}_{\Omega,r})$$

Remark 10. *Notice that theorem 4 gives easily the higher dimensional analog of Lickorish' result. Let L be any parametrized n-sphere in a manifold M^{2n+1} with trivial normal bundle and framing ρ. Surgery on (L, ρ) has an alternative description: the framing furnishes an identification $\nu(L) \cong \underline{\mathbb{R}}^{n+1}{}_{|S^n}$, which is canonically isomorphic to $\underline{\mathbb{R}} \oplus T^*S^n$. The total space of this bundle gives a structure of 2-calibrated foliation in $\nu(L)$, and by proposition 4 lagrangian surgery amounts to attaching an (n+1)-handle to the trivial cobordism along L and with framing ρ. Theorem 4 now implies that surgery on (L, ρ) amounts to cut M open along $\{0\} \times T^*L \subset \nu(L)$ and glue back by a generalized Dehn twist.*

Observe also that according to proposition 6.1 in [6]*, surgeries with framings ρ and $\rho \circ \Delta$ give always diffemorphic manifolds only in dimensions 2 and 6 (see also* [24] *for a sharper result on the period of τ^2 in $\mathrm{Diff}^{\mathrm{comp}}(T^*S^n)$, for n even).*

3.5. **Legendrian surgery, open book decompositions and generalized Dehn surgery.** Let ξ be an exact contact structure on M, and α a 1-form with $\xi = \mathrm{Ker}\alpha$.

Recall that an open book decomposition for M is given by a pair (K, θ) such that

- K is a codimension 2 submanifold with trivial normal bundle, *the binding*,
- $\theta\colon M\backslash K \to S^1$ is a submersion (i.e. a fibration) such that in a trivialization $D^2 \times K$ of a neighborhood of K, it coincides with the angular coordinate.

Let F denote the closure of any fixed leaf. The return map associated to a suitable lift of $\partial/\partial\theta$ to $M\backslash K$, defines a diffeomorphism of F (unique up to isotopy) supported away from a neighborhood of $\partial F = K$.

M -up to diffeomorphism- can be recovered out of F and the return map.

The following discussion is mostly taken from [13]:

Definition 9. *The contact structure ξ is supported by an open book decomposition (K, θ) if for a choice of contact form α we have:*

1. *α restricts to K to a contact form.*
2. *$d\alpha$ restricts to each leaf to an exact symplectic structure.*
3. *The orientation of K as the boundary of each symplectic leaf matches the natural orientation induced by the contact form.*

The form α is then adapted *to the open book (K, θ).*

If α and $f\alpha$, $f \in C^\infty(M)$, $f > 0$ are both adapted to (K, θ), then a fixed leaf inherits different exact and convex at infinity symplectic structures (i.e there exists a Liouville vector field defined in $\mathring{F}$ an transversal near ∂F to its translates associated to any product structure near the boundary), but the completion [10] is unique up to isotopy.

For α a contact form adapted to (K, θ), the Reeb vector field is necessarily tangent to K. Then its flow defines a first return map $\varphi \in \mathrm{Symp}(\mathring{F}, \Omega)$, where Ω is the symplectic structure induced by $d\alpha$ in $\mathring{F}$. Moreover, it is possible to choose $\alpha' = f\alpha$, with $f = 1$ away from any fixed neighborhood of K, so that its monodromy is compactly supported.

Assume that we are now given a closed exact symplectic manifold (F, Ω) convex at infinity (the symplectic structure is also defined in the boundary). Then for any $\varphi \in \mathrm{Symp}^{\mathrm{comp}}(F, \Omega)$, in the obvious closed manifold it is possible to construct a contact structure ξ supported by the obvious open book [40]. Moreover, if we have two contact structures ξ and ξ' supported by the same open book and the inducing on a leaf exact symplectic structures convex at infinity with isotopic completions, then the contact structures are isotopic [12, 13].

Therefore, up to isotopy, (M, ξ) is totally determined by any open book supporting it (i.e. by the completion of the structure of exact symplectic manifold convex at infinity of a leaf, together with the return symplectomorphism which is the identity near the boundary).

The previous result become relevant in light of the following

Theorem 5. [12, 13] *For every exact contact manifold (M, ξ) and any contact form representing α, there exist an open book (K, θ) supporting ξ such that α is adapted to it.*

What is more important Giroux and Mohsen announce that for a fixed contact structure, any two compatible open book of certain class (those coming from

approximately holomorphic geometry constructions) are related by an operation called positive stabilization.

Other result announced in [13] and which has to do with fillability of contact structures is the following: let (M, α) so that the contact form is adapted to the open book (K, θ), and let L be a parametrized legendrian sphere which is contained in a leaf (and hence it becomes lagrangian for the symplectic structure $d\alpha$ in the leaf). Let (M^L, α^L) be the result of performing legendrian contact surgery along L [41].

Notice that away from the binding K, the open book inherits a 2-calibrated structure $(M\backslash K, \mathcal{F}_\theta, d\alpha)$, $\mathcal{F}_\theta = \mathrm{Ker} d\theta$. Then we can perform generalized Dehn surgery along L, obtaining a new open book decomposition (on a new manifold) whose return map is $\tau_L \circ \varphi^{-1}$, with φ the return map associated to (M, α, K, θ) and τ_L a generalized Dehn twist along L.

The result is that the unique contact structure supported by the new open book decomposition is (M^L, α^L).

The ideas developed relating lagrangian surgery and generalized Dehn surgery allow us to give a very natural proof of the aforementioned result.

More precisely, very much as we saw for lagrangian framings in subsection 3.4, given a parametrized legendrian sphere L in (M, α) we have two contact surgeries: we can consider the symplectization of (M, α) and attach a symplectic handle to the convex end to obtain (M^L, α^L). A symplectic handle can also be attached to the concave end obtaining thus a contact manifold (M^{L^-}, α^{L^-}) as the concave end of the corresponding symplectic cobordism (with orientation matching the original one of M in the common piece).

Theorem 6. *Given V any small enough neighborhood of L, with $L \cap K = \emptyset$, there exists an isotopy $\Psi_t \colon M \to M$, $t \in [0, 1]$, starting at the identity and such that*

- *Ψ_t is supported in V and tangent to the identity at L.*
- *$(V, \mathcal{F}_{\theta_t}, d\alpha)$, with $\mathcal{F}_{\theta_t} := \Psi_{t*}\mathcal{F}_\theta$, is a 2-calibrated foliation, and thus the contact form α is adapted to $(K, \Psi_{t*}\theta)$.*
- *The contact form $\alpha^L \in \Omega^1(M^L)$ is adapted to the open book (K, θ_1^L), where $(M\backslash K, \mathcal{F}_{\theta_1^L}, d\alpha^L)$ is (equivalent to) the result of performing generalized Dehn surgery on $(M\backslash K, \mathcal{F}_{\theta_1}, d\alpha)$ along L.*

Proof. We sketch the main ideas.

We consider $(M \times [-1, 1], d(e^t\alpha))$, (a piece of) the symplectization of (M, α). The tuple $(M \times [-1, 1], d(e^t\alpha), \partial/\partial t, M \times \{0\}, L \times \{0\})$ is an isotropic setup in the language of Weinstein [41] (actually the notion of Liouville vector field we use differs from that of Weinstein, for we require its flow to exponentially expand the symplectic form).

The second isotropic setup is the one of the (n+1)-handle to be attached, which is almost the one described in [41]; we change the end along which the handle is glued, and also the Liouville vector field that has to make it concave. We also use the notation of proposition 4.

The symplectic form is $\Omega_{\mathbb{C}^{n+1}}$. Consider the function

$$q = \sum_{i=1}^{n+1} x_i^2 - 2y_i^2$$

Its negative gradient (w.r.t. the Euclidean metric)

$$Z = -2x^1 \frac{\partial}{\partial x^1} + 4y^1 \frac{\partial}{\partial y^1} - \cdots - 2x^{n+1} \frac{\partial}{\partial x^{n+1}} + 4y^{n+1} \frac{\partial}{\partial y^{n+1}}$$

is a Liouville vector field.

For each $r > 0$ consider the hypersurface $q^{-1}(r)$, which contains the lagrangian sphere Σ_r (which is also legendrian w.r.t $\alpha_Z := i_Z \Omega_{\mathbb{C}^{n+1}}$). Let $V(r,\epsilon)$ be the tubular neighborhood or radius $\epsilon > 0$ of Σ_r in $q^{-1}(r)$. Recall that Y_1 is the hamiltonian of $-\mathrm{Re}h$ w.r.t. $\Omega_{\mathbb{C}^{n+1}}$. Notice that $dq(Y_1) < 0$, and therefore Y_1 is transversal to the level hypersurfaces of q.

We claim that for any $\epsilon' > 0$, $\epsilon > \epsilon'$, we have $f_r \in C^\infty(V(r,\epsilon)\backslash\Sigma_r, \mathbb{R}^-)$ a cut-off function with compact support, with

- $\varphi_1^{f_r Y_1}(V(r,\epsilon')\backslash\Sigma_{-r}) \subset q^{-1}(-2r)$
- $\varphi_1^{f_r Y_1}(V(r,\epsilon))$ is transversal to Z.

Once we assume that, we define $M_r^{L,0} := \varphi_1^{f_r Y_1}(V(r,\epsilon)\backslash\Sigma_r) \cup \Sigma_{-r}$. The Liouville vector field Z is transversal to $M_r^{L,0}$, and thus the hypersurface inherits an exact contact structure α_Z.

Our symplectic (n+1)-handle is the compact region bounded by $M_r^{L,0}$ and $V(r,\epsilon)$. The Liouville vector field is Z, the hypersurface $V(r,\epsilon)$ and Σ_r the parametrized legendrian sphere.

The symplectic morphism that defines the legendrian surgery [41] sends $(V(r,\epsilon),\Sigma_r,\alpha_Z)$ to $(\nu(L), L, \alpha)$, so we can consider $(V(r,\epsilon),\Sigma_r,\alpha_Z)$ as a subset of $(\nu(L), L, \alpha)$. Then $M^L := M_r^{L,0} \cup (M\backslash(K \cup V(r,\epsilon)))$ carries and obvious contact structure α^L which restricts to $(M\backslash(K\cup V(r,\epsilon)),\alpha)$.

Notice that since both $M_r^{L,0}$ and $V(r,\epsilon)$ are transversal to Y_1, they inherit 2-calibrated foliations $(M_r^{L,0}, \mathcal{F}_r^L, \omega_r^L)$ and $(V(r,\epsilon), \mathcal{F}_r, d\alpha)$. Theorem 4 easily implies that $(M_r^{L,0}, \mathcal{F}_r^L, \omega_r^L)$ is the generalized Dehn surgery of $(V(r,\epsilon), \mathcal{F}_r, d\alpha)$ along Σ_r.

In $V(r,\epsilon)$ we have two structures of 2-calibrated foliation, $(\mathcal{F}_r, d\alpha)$ and $(\mathcal{F}, d\alpha)$. Since both are *fibrations* transversal to the Reeb vector field , we can use its trajectories to find the desired isotopy (where the neighborhood V in the statement of the theorem will be a small neighborhood of Σ_r contained in $V(r,\epsilon)$).

The claim about the existence of the function f_r is easily proven when $n = 1$, by inspecting the trajectories of both Z and Y. The general case can be reduced to the previous one: each point $(x_1, y_1, \dots, x_{n+1}, y_{n+1})$ in $\mathbb{C}^{n+1}$ and away from the ascending and descending submanifolds and the critical point (these are the same for both morse functions $\mathrm{Re}h$ and q), determines $[x_1 : \cdots : x_{n+1}], [y_1 : \cdots : y_{n+1}]$ a point in $\mathbb{RP}^n \times \mathbb{RP}^n$, which gives rise to two lines in $\mathbb{R}^{n+1}$ and $i\mathbb{R}^{n+1}$ respectively. These lines span a plane in $\mathbb{C}^{n+1} = \mathbb{R} \oplus i\mathbb{R}^{n+1}$. Each plane in the family is preserved by both flows; moreover, they restrict to the planes to the flows of the 1-dimensional case. From this observation the claim follows easily. $\square$

Therefore, if after the small isotopy of the previous theorem we get an open book (that we still denote by (K,θ)) whose monodromy is $\varphi \in \mathrm{Symp}(\mathring{F}, d\alpha)$, then (M^L, α^L) is adapted to an open book with the same symplectic leaf and monodromy $\tau_L \circ \varphi \in \mathrm{Symp}(\mathring{F}, d\alpha)$.

Similarly, if we attach a symplectic handle to the concave end of the symplectization we get the contact manifold (M^{L^-}, α^{L^-}) adapted to an open book whose monodromy is $\varphi \circ \tau_L^{-1}$.

Observe that remark 10 implies that in dimensions 5 and 13 (n=2,6) the manifolds M^L and M^{L^-} are diffeomorphic. In [25] it is shown that there are instances (coming from Brieskorn manifolds) in which (M^L, α^L) and (M^{L^-}, α^{L^-}) are not contactomorphic, and hence the authors can deduce that τ^2 is not isotopic to the identity in $\mathrm{Symp}^{\mathrm{comp}}(T^*S^6, d\alpha_{\mathrm{can}})$, a result already proven by Seidel for n=2 [35] (similar results are also drawn for powers of the Dehn twists known to be isotopic to the identity in $\mathrm{Diff}^{\mathrm{comp}}(T(\lambda))$, for all n even).

For any contact form α representing the given contact structure and L a legendrian submanifold, Giroux and Mohsen announce [13] the existence of relative

open books, i.e. α adapted to the open book and L contained in a leaf (the interested reader familiar with *approximately holomorphic geometry* [7] and its version for contact manifolds [23, 34] can write a proof along the following lines: the open book is the result of pulling back the canonical open book decomposition of $\mathbb{C}$ by an approximately holomorphic function. To make sure the binding does not contain L, one uses reference sections supported in L which achieve the value 1 when restricted to L; they come from an explicit formula once we identify a tubular neighbourhood of L with $(\mathcal{J}^1 L, \alpha_{\mathrm{can}})$. One further adds perturbations whose restriction to L attain real values: they are such that its restriction to $T^*L \times \{0\} \subset \mathcal{J}^1 L$ are small *real multiples* (this is always possible, according to the local perturbation theorem of [2]) of reference sections equivariant w.r.t. the involution on $(\mathcal{J}^1, \alpha_{\mathrm{can}})$ which reverses the sign of the fiber and conjugation on $\mathbb{C}$).

Therefore we conclude that lagrangian surgery includes legendrian surgery, for we can bypass the latter by choosing appropriate compatible open book decompositions and then performing lagrangian surgery. According to theorem 1 we can even claim that generalized Dehn surgery contains legendrian surgery, and forget about the cobordisms.

Actually, the reason why generalized Dehn surgeries for open books supporting the contact structure give the same contact manifold, is because there is a contact surgery behind. Now consider (L, χ) where L is a legendian submanifold of (M^L, α) and $\chi \in \mathrm{Symp}^{\mathrm{comp}}(T^*L, d\alpha_{\mathrm{can}})$; for example we can take L to be the product of two spheres (say a legendrian 2-torus in a contact 5-manifold) and χ be any of the maps induced by a Dehn twist in either of the factor, or its composition. Take any open book relative to L and such that α is adapted to it. Consider the new manifold M^L associated to the open book with symplectic monodromy $\chi \circ \varphi$. It is clear that the diffeomorphism type of the manifold does not depend on the open book, but it is not clear we whether in general the contact structure depends upon the choice of open book. In either case, it would be an interesting situation because it would give either a new contact surgery (possibly a legendrian surgery based on a block different from a symplectic handle), or different contact structures.

4. Lefschetz pencils and symplectic parallel transport

Definition 10. *Let $x \in (M, \mathcal{F}, \omega)$. A chart $\varphi_x \colon (\mathbb{C}^n \times \mathbb{R}, 0) \to (M, x)$ is compatible with $(\mathcal{F}, \omega)$ if it is adapted to $\mathcal{F}$, and $\varphi_x^* \omega$ restricted to the leaf through the origin is of type $(1,1)$ at the origin.*

Definition 11. *(see [34]) A Lefschetz pencil structure for $(M, \mathcal{F}, \omega)$ is a triple (f, B, Δ) where $B \subset M$ is a codimension four 2-calibrated submanifold and $f \colon M \backslash B \to \mathbb{CP}^1$ is a smooth map such that:*

1. *f is a leafwise submersion away from Δ, a 1-dimensional manifold transversal to $\mathcal{F}$ where the restriction of the differential of f to $\mathcal{F}$ vanishes. The fibers of the restriction of f to $M \backslash (B \cup \Delta)$ are 2-calibrated submanifolds.*
2. *Around any point $c \in \Delta$ there exist coordinates $z_1, \dots, z_n, t$ compatible with $(\mathcal{F}, \omega)$, and complex coordinates of $\mathbb{CP}^1$ such that*
$$f(z, t) = z_1^2 + \cdots + z_n^2 + \sigma(t), \tag{43}$$
where $\sigma \in C^\infty(\mathbb{R}, \mathbb{C})$.
3. *Around any point $b \in B$ there exist coordinates $z_1, \dots, z_n, t$ compatible with $(\mathcal{F}, \omega)$, and complex coordinates of $\mathbb{CP}^1$ such that $B \equiv z_1 = z_2 = 0$ and $f(z, t) = z_1 / z_2$.*
4. *$f(\Delta)$ is an immersed curve in general position.*

For each regular value $z \in \mathbb{CP}^1\backslash f(\Delta)$, $f^{-1}(z)$ is an open 2-calibrated submanifold. Its compactification $W_z := f^{-1}(z) \cup B$, a *regular fiber*, is a compact 2-calibrated foliation.

In [29], the following result is proven:

Theorem 7. *Let $(M, \mathcal{F}, \omega)$ an integral 2-calibrated foliation and let h be an integral lift of $[\omega]$. Then for all $k \gg 1$ there exist Lefschetz pencils (f_k, B_k, Δ_k) such that:*

1. *The regular fibers are Poincarè dual to kh.*
2. *The inclusion $l_k \colon W_k \hookrightarrow M$ induces maps $l_{k*} \colon \pi_j(W_k) \to \pi_j(M)$ (resp. $l_{k*} \colon H_j(W_k; \mathbb{Z}) \to H_j(M; \mathbb{Z})$) which are isomorphism for $j \leq n-2$ and epimorphisms for $j = n-1$.*

It is possible to define the blowing up of $(M, \mathcal{F}, w)$ along B as a foliated manifold $(\hat{M}, \mathcal{F})$: one has to use the canonical charts around the points of B to define a leafwise blowing up; in this way we obtain the charts for $\hat{M}$ with an obvious foliation $\mathcal{F}$. The map $f \colon M\backslash B \to \mathbb{CP}^1$ lifts to $\hat{f} \colon (\hat{M}, \mathcal{F}) \to \mathbb{CP}^1$, having as regular fibers the $W_z's$. It is not clear, however, how to endow $(\hat{M}, \mathcal{F})$ with a 2-calibration mimicking the symplectic blowing up.

4.1. **Symplectic parallel transport.** Let (f, B, Δ) be a Lefschetz pencil for $(M, \mathcal{F}, \omega)$. Let $x \in M\backslash(B \cup \Delta)$ and $\mathcal{F}_x$ the leaf through x. Then $W_{f(x)} \cap \mathcal{F}_x$ is a symplectic submanifold of the leaf. Its symplectic orthogonal at x is a (symplectic) plane tangent to $\mathcal{F}_x$. We call the corresponding distribution the *horizontal distribution* and we denote it by $\mathcal{H}$.

Definition 12. *A piecewise smooth curve $\zeta \colon [0,1] \to M\backslash B$ is called horizontal if*

1. *For every $t \in \zeta^{-1}(M\backslash(B \cup \Delta))$, t is a regular point and $\dot{\zeta}(t)$ is tangent to $\mathcal{H}$.*
2. *For every non-regular point $t_0 \in [0,1]$ both limits $\lim_{t \nearrow t_0} \dot{\zeta}(t)$ and $\lim_{t \searrow t_0} \dot{\zeta}(t)$ exist.*

A (piecewise smooth) curve $\gamma \colon [0,1] \to \mathbb{CP}^1$ is compatible with the pencil (f, B, Δ) if its non-regular values are contained in $f(\Delta)$ and for any non-regular point the limits from the left and right of the derivative exist. A horizontal curve $\tilde{\gamma}$ is a horizontal lift for γ if $\gamma = f \circ \tilde{\gamma}$.

We collect a number of result regarding horizontal lifts and parallel transport in the following proposition. They mostly follow from the analogous results for symplectic manifolds which have already been used in section 3.

Proposition 5.

1. *Any horizontal curve contains at most a finite number of critical points.*
2. *For any curve $\gamma \colon [0,1] \to \mathbb{CP}^1$ compatible with the pencil and any $x \in M\backslash(B \cup \Delta)$ with $f(x) = \gamma(0)$, there exist suitable piecewise smooth reparametrizations of γ which posses a horizontal lift starting at x.*
3. *If γ smooth avoids $f(\Delta \cap \mathcal{F}_x)$ and $x \in M\backslash(B \cup \Delta)$ with $f(x) = \gamma(0)$, then $\tilde{\gamma}_x$ is the unique horizontal lift starting at x. Moreover if a lift $\tilde{\gamma}$ avoids Δ and γ is smooth, then it is the unique lift.*
4. *For any smooth curve γ there is a neighborhood U_γ of B in $W_{\gamma(0)}$ such that the (symplectic) parallel transport $\rho_\gamma \colon U_\gamma\backslash B \to W_{\gamma(1)}\backslash B$ is defined. The parallel transport can be smoothly extended to $\rho_\gamma^{\mathrm{ext}} \colon U_\gamma \subset W_{\gamma(0)} \to W_{\gamma(1)}$ by declaring it to be the identity on B. Thus, it lifts to $(\tilde{M}, \mathcal{F})$ preserving the exceptional divisor.*
5. *If $\tilde{\gamma}_x$ is unique then ρ_γ is a local Poisson equivalence around x. For a fixed leaf $\mathcal{F}_o$, if γ smooth curve misses $f(\Delta \cap \mathcal{F}_o)$ then $\rho_\gamma \colon (W_{\gamma(0)}\backslash B) \cap \mathcal{F}_o \to (W_{\gamma(1)}\backslash B) \cap \mathcal{F}_o$ is a symplectomorphism.*

(6) *If γ smooth misses $f(\Delta)$ then $\rho^{\text{ext}}_\gamma\colon W_{\gamma(0)} \to W_{\gamma(1)}$ defines an equivalence of Poisson structures. If $n > 1$ then it is an equivalence of 2-calibrated foliations.*

Proof. Let g be a fixed metric in M and g_{FS} the Fubini-Study metric of $\mathbb{CP}^1$. Let $V^{2n+1}(\delta) \subset \mathbb{C}^n \times \mathbb{R}$ denote $B^{2n}(\delta) \times [-\delta, \delta]$, the product of the balls of radius δ of dimensions $2n$ and 1. Around any point c in Δ there are coordinates $z_1, \ldots, z_n, t$ centered at c defined in $V^{2n+1}(\delta)$, so that $f(z,t) = z_1^2 + \cdots + z_n^2 + \sigma(t)$, with δ and the absolute value of σ and any fixed number of its derivatives bounded independently of c. In $V^{2n+1}(\delta)$ the metric g is comparable to the Euclidean metric (independently of c); also the Euclidean metric can be used in the chart $\mathbb{C}$ of $\mathbb{CP}^1$. We introduce the following notation: $V^{2n+1}(c,\delta) := \varphi_c(V^{2n+1}(\delta))$, $V^{2n}(\delta) := B^{2n}(\delta)$, $V^{2n}(c,\delta) := \varphi_c(V^{2n}(\delta))$

The coordinates, up to a translation in $\mathbb{C}$, restrict to $\mathcal{F}_c$ to usual Morse coordinates. Hence, we deduce:

(a) The critical points of $f_{|\mathcal{F}_c}$ -possibly infinity- are uniformly isolated (w.r.t. the induced metric on the leaf).
(b) There exist $r > 0$ independent of the critical point $c \in \Delta$ such that $f(V^{2n}(c,\delta)) \subset \mathbb{CP}^1$ contains $\bar{D}(f(c), r)$, the ball of radius r centered at $f(c)$ w.r.t. g_{FS}.

Let ζ be a horizontal curve containing $c \in \Delta$. From lemma 10 we deduce that that $V^{2n}(c,\delta) \cap \zeta$ is a piecewise smooth curve whose unique singular point is c; the length of the piece of ζ joining c with $\partial V^{2n}(c,\delta) \cap \zeta$ is bounded by below by a constant $D > 0$. Therefore, two critical points in ζ are at least at distance $2D$, so we have a finite number of them and this proves 1.

Let $\gamma\colon [0,1] \to \mathbb{CP}^1$ compatible with the pencil, $\gamma(0) = z$, and $x \in M \backslash (B \cup \Delta)$. Suppose that the lift $\tilde{\gamma}_x\colon [0, t_0) \to M \backslash (B \cup \Delta)$ converges to Δ. Then lemma 10 implies that $\tilde{\gamma}$ can be extended to $[0, t_0 + \varepsilon]$ (it is perhaps necessary to reparametrize γ to have trivial derivative when it approaches the critical value, so that when $\tilde{\gamma}(t)$ converges to $c \in \Delta$ the derivative does not go to infinity). The lift is a horizontal curve and it is not unique once we cross c (there is an S^n worth of choices).

Suppose that $\tilde{\gamma}\colon [0, t_0) \to M \backslash (B \cup \Delta)$ converges to $b \in B$. Consider the restriction to the leaf of the coordinates around b of definition 11. Then we are in $V^{2n}(\delta) \subset \mathbb{C}^n$ and the fibers are the pencil of hyperplanes with base $B = \{0\} \times \mathbb{C}^{n-2} \subset \mathbb{C}^n$. Away from the base we have $\mathcal{H}$ a smooth distribution of (real) planes. Notice that restricted to each hyperplane of the pencil, the distribution $\mathcal{H}$ extends smoothly to $B \cap V^{2n}(\delta)$, where we get a smooth family of distributions parametrized by $\mathbb{CP}^1$; by compactness the angle of all those 2-planes w.r.t the corresponding hyperplanes is bounded by below by a positive constant. This easily implies that any horizontal curve cannot converge to $B \cap V^{2n}(\delta)$ in finite time. Therefore, horizontal lifts always exist, and they are unique if and only if they miss Δ, which proves point 2 and 3.

Given any embedded curve $\gamma\colon [0,1] \to \mathbb{CP}^1$ it is always possible to find $X \in \mathfrak{X}(\mathbb{CP}^1)$ extending $\dot{\gamma}$. Let $\tilde{X} \in \mathfrak{X}(V^{2n}(\delta))$ defined to be the horizontal lift away from $B \cap V^{2n}(\delta)$, and $\tilde{X}_{|B \cap V^{2n}(\delta)} = 0$. The previous paragraph implies that $\tilde{X}$ is smooth, so its flow defines an isotopy that fixes $B \cap V^{2n}(\delta)$. This isotopy -for the time interval $[0,1]$- restricts to $(W_{\gamma(0)} \backslash B) \cap V^{2n}(b,\delta)$ to the flow associated to the parallel transport over γ, and point 4 follows easily from this.

To prove point 5 we first show that the lift $\tilde{\gamma}_x$ is at positive distance from $\Delta_x := \Delta \cap \mathcal{F}_x$.

Consider

$$\Delta_x(\delta) := \bigcup_{c \in \Delta_x} V^{2n}(c,\delta),$$

where we have coordinates in which $f_{|\mathcal{F}_x}$ has normal form. If the distance of $\tilde{\gamma}_x$ to Δ_x were vanishing then the curve would enter in an infinite number of the previous disjoint "balls" $V^{2n}(c,\delta)$. Select those balls for which $\tilde{\gamma}_x \cap V^{2n}(c,\delta/2) \neq \emptyset$. Hence, on each $V^{2n}(c,\delta)$ the piece of $\tilde{\gamma}_x$ would be of length bigger than some $D' > 0$, but that would contradict the finite length of $\tilde{\gamma}_x$.

Now cover Δ with "balls" $V^{2n+1}(c,\delta)$. By the previous paragraph, even though the leaf $\mathcal{F}_x$ may intersect $V^{2n+1}(c,\delta)$ in an infinite number of plaques, it contains a finite number of connected components of $\tilde{\gamma}_x$. From that we deduce that the distance $d_g(\tilde{\gamma}_x, \Delta)$ is strictly positive. Therefore, we will have a unique lift starting at points in a neighborhood of x in $(W_{\gamma(0)} \backslash B)$. That will define a diffeomorphism which is known to preserve the Poisson structures (for example by lemma 9).

If γ misses $f(\Delta)$, the previous point implies that the (extended) parallel transport $\rho_\gamma^{\text{ext}} \colon W_{\gamma(0)} \to W_{\gamma(1)}$ defines a diffeomorphism which preserves the Poisson structure away from B. Since B is in the closure of the open fiber, then the Poisson structure is preserved in the whole (compact) fiber.

When $n > 1$ the cohomology class can be evaluated on embedded surfaces away from B. Let $\Sigma \subset W_{\gamma(0)} \backslash B$. The parallel transport defines a (trivial) cobordism Q from Σ to $\rho_\gamma(\Sigma)$. Let us call $\omega_{\gamma(0)}$ (resp. $\omega_{\gamma(1)}$) the restriction of ω to $W_{\gamma(0)}$ (resp. $W_{\gamma(1)}$).

$$< [h^{\gamma *}\omega_{\gamma(1)}] - [\omega_{\gamma(0)}], [\Sigma] >= \int_{h^\gamma(\Sigma)} \omega_{\gamma(1)} - \int_\Sigma \omega_{\gamma(0)} = \int_{h^\gamma(\Sigma)} \omega - \int_\Sigma \omega = \int_{\partial Q} \omega = 0,$$

and this finishes the proof of point 6.

□

Let $\mathcal{F}_o$ be a (possibly non-compact) leaf of $(M,\mathcal{F})$. Let us also fix z a regular value of $f_{|\mathcal{F}_o}$. Define $W_{z,o} := W_z \cap \mathcal{F}_o$. This is a smooth submanifold of the leaf.

Let $B_o = B \cap \mathcal{F}_o$, $W^*_{z,o} := W_{z,o} \backslash B_o$. The submanifold B_o has codimension 2 in $W_{z,o}$. Thus, $W_{z,o}$ is connected if and only if $W^*_{z,o}$ is connected. We will prove the connectedness of the latter.

Select $W^{*\text{con}}_{z,o}$ a connected component of $W^*_{z,o}$. Let $z' \neq z$ any other regular value for $f_{|\mathcal{F}_o}$. Let γ be a smooth curve joining z and z' and avoiding $f(\Delta_o)$. According to point 5 in proposition 5, $\rho_\gamma(W^{*\text{con}}_{z,o})$ is a submanifold of $W^*_{z',o}$. Moreover it is a connected component for if this were not the case, we could reverse the transport and conclude that $W^{*\text{con}}_{z,o}$ is not a connected component of $W^*_{z,o}$.

Proposition 6. *Let γ' be another smooth curve joining z and z' and avoiding $f(\Delta_o)$. Then $\rho_\gamma(W^{*\text{con}}_{z,o}) = \rho_{\gamma'}(W^{*\text{con}}_{z,o})$.*

Proof. Case 1. Let γ' be such that for some $x \in W^{*\text{con}}_{z,o}$ and $c \in \Delta_o$, the curves $\tilde{\gamma}_x$ and $\tilde{\gamma}'_x$ only differ inside $V^{2n}(c,\delta)$.

Recall that $f_{|\mathcal{F}_o}(V^{2n}(c,\delta/2))$ contains $\bar{D}(f(c),r)$; also the intersection of each regular fiber of $f_{|\mathcal{F}_o}$ in $V^{2n}(c,\delta)$ is connected. If we work $V^{2n}(\delta)$ then the parallel transport is determined entirely by the 2-form. If r is taken small enough the symplectic form does not differ much from its restriction to the origin. Hence, by compactness of Δ, there exists $l(r) > 0$ independent of c such that for curves $\sigma \subset \bar{D}(f(c),r)$ of length bounded by $l(r)$ and $y \in V^{2n}(c,\delta/2)$, we have $\tilde{\sigma}_y(1) \in V^{2n}(c,\delta)$.

Let us assume that $z' \in \bar{D}(f(c),r)$. Let $t_0 \in [0,1]$ such that (i) $\gamma([t_0,1]) \subset \bar{D}(f(c),r)$ has length bounded by $l(r)$ and (ii) $\tilde{\gamma}_x(t_0) \in V^{2n}(c,\delta/2)$. Let γ' be any smooth curve connecting z and z' avoiding $f(\Delta_o)$, such that

- $\gamma'_{|[0,t_0]} = \gamma_{|[0,t_0]}$.
- $\gamma'_{|[t_0,1]} \subset \bar{D}(f(c),r)$, and the length of $\gamma'_{|[t_0,1]}$ is bounded by $l(r)$.

The previous discussion implies that $\tilde{\gamma}'_x$ is unique and $\tilde{\gamma}_x(1), \tilde{\gamma}'_x(1) \in V^{2n}(c,\delta)$.

Both $\rho_\gamma(W^{*\mathrm{con}}_{z,o})$ and $\rho_{\gamma'}(W^{*\mathrm{con}}_{z,o})$ are smooth connected components of $W^*_{z',o}$, and both contain the unique connected component of $W_{z',o} \cap V^{2n}(c,\delta)$. Hence they are the same component.

Case 2. Let γ' be such that for some $x \in W^{*\mathrm{con}}_{z,o}$ and $c \in \Delta_o$, the curves $\tilde{\gamma}_x$ and $\tilde{\gamma}'_x$ only differ away from $\Delta_o(\delta/2)$.

Let $B(\delta)$ denote the tubular neighborhood of B of radius δ, and $B_o(\delta)$ its intersection with $\mathcal{F}_o$. Using compactness of B and the models furnished by the charts of definition 11 centered at points of b, it is easy to conclude that the connectedness of $W^*_{z,o}$ is equivalent to the connectedness of $W_{z,o} \backslash B_o(\delta)$, for $\delta > 0$ small enough.

Let $G(\delta) := M \backslash (\Delta(\delta/2) \cup B(\delta))$, $G_o(\delta) := G(\delta) \cap \mathcal{F}_o$. Compactness of $G(\delta)$ implies that in the points of $G_o(\delta)$ the derivative of $f_{|\mathcal{F}_o}$ is bounded by below, and that at any $p \in G_o(\delta)$ we can find charts

$$\varphi_p \colon (V^{2n}(\delta), 0) \subset \mathbb{C}^n \to (\mathcal{F}_o \backslash G(\delta), p)$$

with the following properties:

(1) The induced metric can be compared by the Euclidean metric.
(2) In the coordinates $z_1, \ldots, z_n$ of $\mathbb{C}^n$ and one of the canonical affine coordinates of $\mathbb{CP}^1$, the map f is the projection onto the first coordinate up to a translation.
(3) The angle between the symplectic horizontal distribution (w.r.t. $\Omega_{\mathbb{C}^{n+1}}$) and the one associated to the Euclidean metric is bounded by above by a constant C, $|C| < \pi/2$.

Notice that for every $\delta > 0$ small enough there exists $r'(\delta) > 0$, such that the image of $V^{2n}(p,\delta/2)$ also contains the disk $\bar{D}(f(p), r')$. Similarly, there exists $l'(r')$ such that if the length of $\sigma \subset \bar{D}(f(p), r')$ is bounded by $l'(r')$, then for any $y \in V^{2n}(p,\delta/2)$ we conclude that $\tilde{\sigma}_y(1) \in V^{2n}(\delta)$.

Let γ, γ' be to smooth curves in the complement of $f(\Delta_o)$ such that for some $t_0 < t_1$ in $[0,1]$,

- $\gamma_{|[0,t_0]} = \gamma'_{[0,t_0]}$, $\gamma_{|[t_1,1]} = \gamma'_{[t_1,1]}$.
- There exists $p \in G_o(\delta)$ such that $\gamma_{|[t_0,t_1]}$ and $\gamma_{[t_0,1]}$ have length bounded by l' and are contained $\bar{D}(f(p), r')$.
- $\tilde{\gamma}_x(t_0) \in V^{2n}(p, \delta/2)$.

Then it follows that $\rho_\gamma(W^{*\mathrm{con}}_{z,o}) = \rho_{\gamma'}(W^{*\mathrm{con}}_{z,o})$.

We can reduce the general case to the two already studied. Let γ, γ' be two smooth curves joining z and z' and avoiding $f(\Delta_o)$. Choose a finite family of smooth curves $\gamma_0 = \gamma, \ldots, \gamma_m = \gamma'$ avoiding $f(\Delta_o)$ such that for each consecutive pair γ_j, γ_{j+1}, the curves coincide but for arcs σ_j, σ_{j+1} of length bounded by $\min(l, l')$, and for any $z'' \in \sigma \cup \sigma'$, we have $\sigma \cup \sigma' \subset \bar{D}(z'', \min(r, r'))$.

If we apply induction on j, one checks that each step reduces to either case 1 or case 2. Therefore, $\rho_\gamma(W^{*\mathrm{con}}_{z,o}) = \rho_{\gamma'}(W^{*\mathrm{con}}_{z,o})$ and the transport does not depend on the curve. □

Theorem 8. *Let $(M, \mathcal{F}, \omega)$ be a 2-calibrated foliation, dim$M \geq 5$, and let (f, B, Δ) be a Lefschetz pencil as in theorem 7. Then any smooth fiber W of the pencil intersects every leaf in a unique connected component*

Proof. Let $\mathcal{F}_c$ be a compact leaf. Then restriction of (f, B, Δ) to $\mathcal{F}_c$ is a Donaldson Lefschetz pencil [8]. It comes with an associated real Morse function the index of whose critical points is known. The dimension hypothesis implies that $W \cap \mathcal{F}_c$ is connected.

Let $\mathcal{F}_o$ be a non-compact leaf. The restriction of (f, B, Δ) to $\mathcal{F}_o$ defines a function to $\mathbb{CP}^1$ which is a submersion away from a (possibly infinite) set of isolated critical points, for which we have complex charts in which the function is the unique non-degenerate complex quadratic form. This "pencil" does not come with a Lefschetz hyperplane theorem, for the associated real Morse function is not proper.

Let $\Gamma_{z,o}^{\text{con}}$ be the set of horizontal curves ζ starting at $W_{z,o}^{*\text{con}}$ and whose image $f \circ \zeta$ is immersed (away from $t \in [0,1]$ with $\zeta(t) \in \Delta$).

We define

$$\mathcal{F}_{z,o}^{*\text{con}} := \{y \in \mathcal{F}_o \backslash B_o \mid \exists\, \zeta \in \Gamma_{z,o}^{\text{con}},\, \zeta(1) = y\}$$

By construction $\mathcal{F}_{z,o}^{*\text{con}}$ is non-empty (and connected). We want to show that it is open for in that case if there is more than one connected component in $W_{z,o}^{*}$, then $\mathcal{F}_o \backslash B_o$ –itself connected– would be the disjoint union of more than one open non-trivial subset, which is impossible.

Let $y \in \mathcal{F}_{z,o}^{*\text{con}}$ such that the horizontal curve ζ connecting the point with $W_{z,o}^{*\text{con}}$ avoids the critical points. We can suppose that without loss of generality that $\gamma := f \circ \zeta$ is embedded, for otherwise we only need to apply a finite number of times the proof for $f \circ \zeta$ embedded. That y is open follows essentially from point 5 in proposition 5: we select

$$F\colon [-1,1] \times [-\varepsilon, 1+\varepsilon] \to \mathbb{CP}^1 \tag{44}$$

a diffeomorphism with image U, and such that $F_{|\{0\}\times[0,1]} = \gamma$. If t is the coordinate of the interval $[-\varepsilon, 1+\varepsilon]$, consider

$$\frac{d}{dt} F \in \mathfrak{X}(U)$$

and denote its horizontal lift by Z, defined in a domain $\tilde{U}$. Since an integral curve of Z contains ζ, and Z is defined in a tubular neighborhood of ζ, its flow defines a local diffeomorphism from V a small neighborhood of y. By the local model around x and y (both belong to $G_o(\delta)$, for some $\delta > 0$), we conclude that all the integral (horizontal) curves ζ_p, $p \in V$, intersect $W_{z,o}^{*\text{con}}$ (perhaps prolonging them a little bit). Notice also that each $f \circ \zeta_p$ is an embedded curve.

Let $c \in \mathcal{F}_{z,o}^{*\text{con}}$ be a critical point, and let ζ be the connecting horizontal curve; we suppose that $\zeta \cap \Delta = \{c\}$. Assume again that $\gamma := f \circ \zeta$ is embedded. From the lnormal form around c for $f_{|\mathcal{F}}$ and the results of subsection 3.4, we know that there is a lagrangian sphere $L \subset W_{z,o}^{*\text{con}}$ characterized as the set of points in $x \in W_{z,o}^{*\text{con}}$ such that $\tilde{\gamma}_x(1) = c$. Fix F as in equation 44 and let Z be the horizontal lift of d/dtF (away from critical points). If we take $\lambda > 0$ small enough, then the integral curves of Z starting at $T(\lambda) \subset W_{z,o}^{*\text{con}}$ sweep out a tubular neighborhood of c. Both the assumption of $f \circ \zeta$ and c being the only critical point in ζ imply no restriction.

Using the ideas of proposition 6 we can show that for any $z' \notin f(\Delta_o)$, $\mathcal{F}_{z,o}^{*\text{con}} \cap f^{-1}(z') = \rho_\gamma(W_{z,o}^{*\text{con}})$, for any immersed curve $\gamma \subset \mathbb{CP}^1 \backslash f(\Delta_o)$ joining z and z'. □

Corollary 1. *Let W be a smooth fiber of a Lefschetz pencil for $(M, \mathcal{F}, \omega)$, $dimM \geq 5$. Then the inclusion $l\colon (W, \mathcal{F}_W) \to (M, \mathcal{F})$ descends to a homeomorphism*

$$\eta\colon W/\mathcal{F}_W \to M/\mathcal{F}$$

Proof. This is point 1 in lemma 2 applied to $l\colon (W, \mathcal{F}_W) \to (M, \mathcal{F})$. The first necessary hypothesis is l being transversal to $\mathcal{F}$, which holds trivially. The second is the intersection of W with every leaf of $\mathcal{F}$ being connected, and it is the content of proposition 8. □

Proof of theorem 1. Let $(M, \mathcal{F}, \omega)$ be a 2-calibrated foliation. If it is not integral, the compactness of M implies that we can slightly modify ω into ω' so that a suitable multiple $k\omega'$ defines an integral homology class. Theorem 7 implies the

existence of a Lefschetz pencil (f, B, Δ). Let $(W, \mathcal{F}_W, k\omega'_{|W})$ be a regular fiber. Corollary 1 implies the inclusion induces a homeomorphism

$$\eta\colon W/\mathcal{F}_W \to M/\mathcal{F}$$

If the dimension of W is bigger than 3, we apply the same construction to $(W, \mathcal{F}_W, k\omega'_{|W})$. By induction, we end up with a 3-dimensional taut foliation such that $(W^3, \mathcal{F}_W) \hookrightarrow (M, \mathcal{F}, \omega)$ descends to a homeomorphism of leaf spaces. □

4.2. **Regular fibers and lagrangian surgery.** Theorems 7 and 8 describe part of the homology (resp. homotopy) of the regular fibers, and of the topology of their leaf spaces in terms of the corresponding data for $(M, \mathcal{F})$.

We want to understand how different regular fibers are related.

If two regular values z and z' belong to the same connected component of $\mathbb{CP}^1 \backslash f(\Delta)$, then for any curve γ in that connected component connecting z with z', point 6 in proposition 5 implies that $\rho_\gamma^{\text{ext}}\colon W_z \to W_{z'}$ is an equivalence of 2-calibrated structures if $n > 2$, and of Poisson structures if $n = 2$ (recall that the fibers have dimension 2n-1).

We notice that any two arbitrary regular values z and z' can always be joined by a curve γ transversal to $f(\Delta)$.

Theorem 9. *(see [34]) Let $z, z' \in \mathbb{CP}^1$ be two regular values. Let γ be a curve joining z and z' and transversal to $f(\Delta)$. Then $\tilde{f}$ is a cobordism between both fibers which amounts to add one n-handle for each point $x \in \Delta$ such that $f(x) \subset \gamma$. More precisely, if $n > 2$ and there is only one critical point in $f^{-1}(\gamma)$, then there exists $L \subset W_z \backslash B$ a framed lagrangian sphere such that $W_{z'}$ is the result of performing generalized Dehn surgery on W_z along L.*

Proof. Strictly speaking, the cobordism is $\tilde{f}^{-1}(\gamma)$ and occurs in the blown up manifold, but since the handles are attached in arbitrarily small neighborhoods of the critical points, we can equally work on M.

Let $w \in \gamma$ and $c \in \Delta$ with $f(c) = w$. Let us fix adapted charts $\varphi_c\colon V^{2n+1}(\delta) \subset \mathbb{C}^n \times \mathbb{R} \to V^{2n+1}(c, \delta)$ and a canonical affine chart of $\mathbb{CP}^1$ centered at c and w respectively, such that equation 43 holds. We fix $r_0 > 0$ such that $\bar{D}(r_0) \subset f(V^{2n+1}(\delta/2))$ (we omit the pullback of f by the chart φ_c in the notation).

From now on we will work in the adapted coordinates furnishes by φ_c and use the notation of section 3.

We assume also that $z = (r_0, 0)$, $z' = (-r, 0)$ and $\gamma = h_0(r, -r)$, i.e. the real segment form r to $-r$.

Since f is transversal to γ, $f^{-1}(\gamma) \cap V^{2n+1}(\delta)$ is a manifold with corners.

The proof of $Z(\delta) := f^{-1}(\gamma) \cap V^{2n+1}(\delta)$ being a cobordism with attaching sphere Σ_r, $r \in (r, r_0)$, is left for the interested reader.

Let $W_r(\delta) := W_r \cap Z(\delta)$, $r \in [-r_0, r_0]$.

We will adapt theorem 4 to our new setting and define for all $r > 0$ small enough an equivalence of 2-calibrated foliations

$$\phi_r\colon W_r \to W_{-r}$$

Stage 1 is the same. We define

$$\phi_r := \rho^{\text{ext}}_{\gamma|[-r,r]}\colon W_r \backslash W_r(\delta/2) \to W_{-r} \tag{45}$$

To carry out the remaining steps we need to describe the relation of f and ω with the projection $p_1\colon \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}^n$.

The tangent space of $Z(\delta)$ at the origin is the leaf through the origin. Therefore the projection p_1 restricted to $Z(\delta)$ is a diffeomorphism.

Recall that

$$\begin{array}{rcl} h\colon \mathbb{C}^n & \longrightarrow & \mathbb{C} \\ (z_1,\ldots,z_n) & \longmapsto & {z_1}^2+\cdots+{z_n}^2 \end{array}$$

Let $\sigma_r := (0,r)+\sigma$. The leaves of $V^{2n+1}(\delta/2)$ are parametrized by $t\in[-\delta/2,\delta/2]$. The restriction of f to each leaf is $h_t := h+\sigma(t)$. Therefore,

$$W_r(\delta) = \bigcup_{t\in[-\delta,\delta]} h^{-1}(r+\sigma(s))$$

Let Ω_t, $t\in[-\delta/2,\delta/2]$ be the restriction of ω to the corresponding t-leaf. The symplectic forms $\Omega_t\in\Omega^2(V^{2n}(\delta/2))$ are all compatible with h and hence with h_t. They are not a constant family in general. Let Y_t be the hamiltonian w.r.t Ω_t of $-\mathrm{Im}h_t$ (or $-\mathrm{Im}h$).

Let $h(\sigma_r(t))$ be the horizontal segment joining $\sigma_r(t)$ with $\sigma_{-r}(t)$. The Poisson morphism $\rho^{\mathrm{ext}}_{\gamma|[-r,r]}$, when restricted to the t-leaf is $\rho_{\Omega_t,h(\sigma_r(t))}$.

In order to make things more similar to the constructions of theorems 3 and 4, we will assume $\sigma(t)=(0,t)$. This is no restriction, for we can rescale the y coordinate on $\mathbb{R}^2$ so that the $\sigma_2(t)=t$ (recall that diffeomorphisms on $(\mathbb{R}^2,0)$ do not alter the results of the aforementioned theorems). Then on each t-leaf the parallel transport (the flow of Y_t) over the horizontal line through $(0,t)$ can be used to construct a Poisson equivalence in $V^{2n+1}(\delta)$ between $f^{-1}(r)$ and $(h+t)^{-1}(r)$.

Consider $R'_r\in\mathfrak{X}(W_r(\delta))$ the vector field in the kernel of $\omega_{|W_r(\delta)}$ whose flow for time t sends the t'-leaf to the $t'+t$-leaf.

Fix an identification from $(T(\lambda),d\alpha_{\mathrm{can}})$ with $(U,\Sigma_{\Omega_0,r})$ and use $\rho_{\Omega_0,h_0(r,-r)}$ and the flow of R'_r to construct the sets $T_r(\lambda,\epsilon)\subset W_r(\delta)$. We also have the subsets

$$F_r(\lambda,\lambda',\epsilon,0)\subset V^{2n+1}(\epsilon) := \bigcup_{l\in[-r,r]} \rho_{\Omega_t,h_t(r,l)}(A_r(\lambda,\lambda',\epsilon,0))$$

of lemma 8, whose closure does not contain the origin (here the t-leaf of $A_r(\lambda,\lambda',\epsilon,0)$) is translated w.r.t. Ω_t over the segment $h_t(r,l)$). Therefore, we can equally use the metric g_R in $A_r(\lambda,\lambda',\epsilon,0)$ instead of the Euclidean metric.

Since we are working in a contractible neighborhood of the critical point, we have $\omega=d\alpha$. Let α_t denote the restriction of α to each t-leaf of $V^{2n+1}(\delta/2)$, so $d_{\mathcal{F}}\alpha_t=\Omega_t$. The exact structure we use in $T_r(\lambda,\epsilon)$ is the restriction of α_t to the corresponding t-leaf.

Stage 2 in theorem 4 amounts to interpolate for each r small enough between ϕ_r, and a map $\tilde{\phi}_r$ the map which coincides with ϕ_r for t-leaves $\notin[-\epsilon_r,\epsilon_r]$, and whose expression for t-leaves in $[-\epsilon_r,\epsilon_r]$ is given in equation 42.

In the current situation ϕ_r is given by the family of symplectomorphisms

$$\varphi^{R'_r}_{t-\kappa_r(t)}\circ\rho_{\Omega_{\kappa_r(t)},h_{\kappa_r(t)}(r,-r)}\circ\varphi^{R'_r}_{\kappa_r(t)-t}$$

that incorporate the dependence on t of the symplectic forms Ω_t.

Recall that the symplectic form Ω_0 is Kahler at the origin. Let Ω'_l, $l\in[0,1]$ be a family of such forms such that $\Omega'_1=\Omega_0$, $\Omega'_0=\Omega_{\mathbb{C}^{n+1}}+d\zeta$. Let $\beta_r\colon[-\delta/2,\delta/2]\to[0,1]$ is such that $\beta_r(-t)=\beta_r(t)$, $\beta_{r|[0,\epsilon_r/2]}=0$, $\beta_{r|2\epsilon_r/3,\delta/2]}=1$. We define $\tilde{\phi}_r$ by the formula

$$\varphi^{R'_r}_{t-\kappa_r(t)}\circ\varphi^{-1}_{1,r}\circ\varphi_{\beta_r(t),r}\circ\rho_{\Omega'_{\beta_r(t),h_{\kappa_r(t)}(r,-r)}}\circ\varphi^{-1}_{\beta_r(t),r}\circ\varphi_{1,r}\circ\varphi^{R'_r}_{\kappa_r(t)-t}, \tag{46}$$

In order to connect ϕ_r and $\tilde{\phi}_r$ by a family $\phi_{r,s}$ we just need to use a the family of functions $\kappa_{r,s}$ and $\beta_{r,s}$ in stage 2 in the proofs of theorems 3 and 4 respectively.

The same arguments used in stage 2 in theorem 4 show that these maps are exact symplectomorphisms, and that for all $r > 0$ small enough we can apply remark 7 to get the desired exact Poisson morphism.

Stage 3 works exactly as in theorem 4, but also incorporating the dependence on t on the symplectic forms for all leaves. Step 4 is the same.

□

4.3. **Growth properties of the leaves.** It follows from point 1 in lemma 2 that (i) if a leaf $\mathcal{F}_x$ of M has polynomial growth then its intersection with a regular fiber of a pencil W_z has also polynomial growth and (ii) if the intersection $\mathcal{F}_x \cap W$ is compact, since W intersects every fiber of $\mathcal{F}$, then $\mathcal{F}_x$ has to be compact.

We do not know whether polynomial growth of $\mathcal{F}_x \cap W$ implies polynomial growth of $\mathcal{F}_x$.

Let α be a 1-form defining $\mathcal{F}$. Then there exists a 1-form η such that $d\alpha = \eta \wedge \alpha$. The 3-form $\eta \wedge d\eta$ is closed. The Gobdillon-Vey class is defined to be $[\eta \wedge d\eta] \in H^3(M; \mathbb{R})$ and it is well defined. In dimension 3 the evaluation of the Godbillon-Vey class on the fundamental class of the manifold is called the Godbillon-Vey number.

Assume that M has dimension 5. Let (f_k, B_k, Δ_k) be a Lefschetz pencil and $l_k : W_k \hookrightarrow M$ a regular fiber Poincarè dual to kh, where h is an integral lift of $[\omega]$. Then it can be easily checked that Godbillon-Vey number of W_k is $k \int_M \omega \wedge \eta \wedge d\eta$.

5. Lefschetz pencils and harmonic measures

A foliation can be understood as a generalization of a dynamical system, the leaves being the analogs of the orbits. An appropriate tool to study the ergodic theory of a foliation is a transversal invariant measure, but these do not always exist. In [11] Garnett introduced the so called harmonic measures. Those generalize the invariant ones and do always exist. Besides, they are suited to develop an ergodic theory.

Given a leafwise metric $g_{\mathcal{F}}$, let Δ_g denote its leafwise laplacian.

Definition 13. *A probability measure m is harmonic (w.r.t $g_{\mathcal{F}}$) if*

$$\int_M \Delta_g \psi m = 0,$$

for every ψ continuous and C^2 along the leaves and such that $\Delta_g \psi$ is continuous.

Harmonic measures always exist. Locally, in a foliated chart, a harmonic measure decomposes into a transversal measure and the leafwise measure associated to $g_{\mathcal{F}}$ multiplied by a harmonic function [11].

Those results hold for any foliated manifold. The existence of a harmonic measure follows from the ellipticity of the leafwise laplacian together with the Hahn-Banach theorem (an alternative proof uses a fixed point theorem and deep results on the process of diffusion along the leaves [5]).

We want to give a geometric proof for 3-dimensional taut foliations.

Let $(M^3, \mathcal{F})$ a taut foliation and (f, Δ) a Lefschetz pencil (B is empty). The complement of $f(\Delta) \subset \mathbb{CP}^1$ is a collection of 2-cells $D_1, \ldots, D_p$. The inverse image of each cell is a collection of open solid tori T_{ij}, $1 \leq i \leq p$, $1 \leq j \leq l_p$. For each tori we choose a fiber and a diffeomorphism from it to S^1 parametrized by $\theta \in [0, 2\pi]$. We extend it to a map

$$\tau_{ij} : T_{ij} \to S^1, \tag{47}$$

by requiring it to be leafwise constant.

Let ν_g denote the leafwise area form. We define on T_{ij} the volume form

$$\nu_g \wedge \tau_{ij}^* d\theta$$

which is well defined even though ν_g is a foliated area form.

Definition 14. *The measure associated to the pencil* (f, Δ) *and diffeomorphisms* l_{ij} *is*

$$m(\psi) := \sum_{ij} \int_{T_{ij}} \psi \nu_g \wedge \tau_{ij}^* d\theta \tag{48}$$

Proposition 7. *The normalization of the measure of definition 14 is a harmonic measure whose support is* M*.*

Proof. The complement of $\bigcup T_{ij}$ is the disjoint union of Δ and an open surface, and hence has measure zero (w.r.t. any riemannian metric). Therefore the support of m is M.

The harmonicity is straightforward.

We need to show that $m(T_{ij}) < \infty$. We will prove the existence of a constant C such that for any leaf $\mathcal{D}_\theta$ of T_{ij} (diffeomorphic to an open disk),

$$\int_{\mathcal{D}_\theta} \nu_g \leq C$$

Hence

$$\int_{T_{ij}} \nu_g \wedge \tau_{ij}^* d\theta = \int_{S^1} (\int_{\mathcal{D}_\theta} \nu_g) d\theta \leq 2\pi C$$

To bound the surface of the disk $\mathcal{D}_\theta$ we fix U a tubular neighborhood of Δ covered by a finite number of adapted charts centered at points of Δ. On each chart we have coordinates z, t with domain of the form $B^2(\delta) \times [-\delta, \delta]$, and $f(z,t) = z^2 + \sigma(t)$.

Let $\mathcal{D}_\theta^0$ be $\mathcal{D}_\theta \cap U$ and $\mathcal{D}_\theta^1 = \mathcal{D}_\theta \backslash \mathcal{D}_\theta^0$.

Let us fix the round metric on $\mathbb{CP}^1$ and let ν_1 denote its area form. In the complement of U the leafwise derivative of f is bounded by below by $\eta > 0$. Therefore,

$$\int_{\mathcal{D}_\theta^1} \nu_g \leq \int_{D_i} 1/\eta \nu_1 \leq 4\pi/\eta$$

Let $c \in \Delta$ be a point in $\tilde{\mathcal{D}_\theta} \backslash \mathcal{D}_\theta$. In the compatible chart with coordinates z, t it goes to a point $(0, t_\theta)$ in the vertical axis. In the cylinder $B^2(\delta) \times [-\delta, \delta]$ the inverse image of $f(\Delta)$ is diffeomorphic to the union of the two coordinate planes containing the vertical axis; the diffeomorphism is the identity when $\sigma(t) = t$. The images of the planes are still transversal to the complex disks, and therefore they split each $B^2(t, \delta)$ of $B^2(\delta) \times [-\delta, \delta]$ centered at $(0, t)$ in four components, two of which map to D_i.

Clearly, the area of $\mathcal{D}_\theta^0 \cap B^2(t_\theta, \delta)$ is bounded by above by the area of $B^2(t_\theta, \delta)$; by compactness there is a common bound A for that area independently of the leaf $B^2_{t_\theta,\delta}$. The relevant observation is that the local model implies that the area of $\mathcal{D}_\theta^0 \cap B^2(t_x, \delta)$ is bounded from below (independently of the disk $\mathcal{D}_\theta$). Therefore, $\mathcal{D}_\theta$ only accumulates into at most $N < \infty$ points of Δ. Hence, the area of $\mathcal{D}_\theta^0$ is bounded by NA.

□

Remark 11. *For simplicity we have worked in the smooth category. Theorem 7 and all the derived results hold also for* C^3 *foliated manifolds.*

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